The Economics of Monotonicity Conditions: Exploring Choice Incentives in IV Models

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Abstract

This paper investigates the role of economic incentives in identifying treatment effects within instrumental variable (IV) models featuring multiple choices. We introduce a comprehensive framework that leverages revealed preference analysis to convert choice incentives into identification conditions. Our analysis reveals that widely-used identification strategies, which depend on monotonicity conditions, are fundamentally linked to specific properties of choice incentives. Furthermore, we uncover novel identification strategies that emerge when individuals encounter non-standard choice incentives. We use this framework to assess the choice incentives of key studies in policy evaluation. We apply the framework to study the impact of education on migration decisions among low-income Mexican families. Utilizing data from the Oportunidades program and recent machine learning techniques, we estimate the causal effect of schooling on migration. Our findings indicate that completing middle school significantly increases the likelihood of migration, while additional education beyond middle school does not. These results contribute to the literature suggesting a non-monotonic relationship between educational attainment and migration decisions among disadvantaged Mexican households.
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1 Introduction

Instrumental variables (IV) are widely used in economics to evaluate the causal effect of an endogenous treatment on an outcome of interest. A valid IV is an exogenous variable that affects the outcome only through its impact on the treatment choice. However, identifying causal effects in IV models requires additional assumptions beyond the existence of a valid instrument. The traditional approach is to impose choice restrictions that constrain how individuals select treatments in response to changes in the instrumental variable.

A seminal identification assumption in IV models with a binary choice is the *monotonicity condition* of Imbens and Angrist (1994). The condition asserts that a change in the instrument must affect all agents in the same direction, either by encouraging them to adopt or rejecting the treatment. Vytlacil (2002) demonstrates that this monotonicity condition is equivalent to assuming a *separability condition*, in which the treatment choice is an indicator function that compares the propensity score with a latent variable that affects the outcome. (Heckman and Vytlacil, 1999) and Heckman and Vytlacil (2005) greatly explore the properties choice models that employ the separability condition. These pioneering concepts spiked a vast literature on both the empirical and theoretical aspects of identification assumptions in IV models with binary choices.[1]

The IV literature has successfully expanded the concepts of monotonicity and separability to multiple-choice models. Angrist and Imbens (1995) extends the monotonicity condition of Imbens and Angrist (1994) from binary to multiple choice models. Their condition is often used to investigate treatment variables that possess a natural order. Ordered choice models are also examined by Cameron and Heckman (1998) and further studied by Carneiro et al. (2003) and Cunha et al. (2007). A significant advance in the IV literature of multiple choice models is due to Lee and Salanié (2018). They develop general identification results for choice models characterized by coherent separability conditions. Heckman and Pinto (2018), on the other hand, propose the unordered monotonicity condition that applies to treatment choices that are not ordered.[2] More recently, Rose and Shem-Tov (2021) propose a monotonicity condition called extensive margin compliers only (EMCO), wherein a change in the instrument incentivizes all agents to shift from no treatment to some treatment status. Mogstad et al. (2021a,b) investigate the monotonicity criteria in a choice model with multiple instrumental variables.

We propose a departure from the traditional approach that governs the identification analysis of IV models. Rather than focusing on novel monotonicity or separability conditions, we explore how choice incentives and classical economic behavior can be leveraged to generate identification

[1]For examples of works in this literature, see Aliprantis (2012); Angrist et al. (2000); Barua and Lang (2016); Dahl et al. (2017); de Chaisemartin (2017); Heckman (2010); Heckman and Urzúa (2010); Heckman and Vytlacil (2007a,b); Huber et al. (2017); Huber and Mellace (2012, 2015); Hull (2018); Imbens and Rubin (1997); Klein (2010); Mogstad et al. (2018); Mogstad and Torgovitsky (2018); Small and Tan (2007).

[2]Unordered choice models have been studied mainly through the literature on structural equations. A common approach assumes that additively separable threshold-crossing models generate the equations that govern the treatment. Examples of this literature are Heckman et al. (2006, 2008); Heckman and Vytlacil (2007a,b).

conditions in IV models with multiple choices and categorical instruments. Our approach is founded on a comprehensible framework that employs revealed preference analysis to transform choice incentives into identification conditions. We demonstrate that the identification assumptions commonly employed in the IV literature can be traced back to specific properties of choice incentives. Moreover, we show that our framework is a valuable tool for developing new identification strategies grounded in economic behaviors.

A benefit of our framework is that identification does not rely on statistical or functional form assumptions. Instead, identification conditions arise from applying fundamental principles of economic behavior to choice incentives. This feature enhances the credibility and comprehension of the identification mechanism. The method is flexible enough to cover a broad spectrum of non-trivial identification assumptions. We demonstrate its versatility by analyzing well-established examples of choice incentives in the policy evaluation literature. Additionally, the framework can offer novel solutions to non-standard economic scenarios where the identification assumptions typically invoked by the IV literature are seldom justified.

We use our framework to investigate the migration of poor Mexican households to the US. Currently, the US is home to approximately 12 million undocumented residents, with almost half of these migrants originating from Mexico. Borjas (1987, 1994) suggest an adverse selection in migration patterns, with lower-skilled workers benefiting the most from moving to the US. This perspective is supported by Angelucci (2015), who shows that schooling incentives led lower-skilled migrants to move to the US. She uses data from Oportunidades, Mexico's flagship anti-poverty program (Gertler, 2004).

On the other hand, Behrman et al. (2005); Chiquiar and Hanson (2005); Hanson (2006) posit a non-monotonic relationship between education and migration. They argue that fundamental skills, such as basic English proficiency acquired in middle school, increase the likelihood of migration. Conversely, further education diminishes the propensity to migrate by enhancing the attractiveness of the domestic labor market relative to its international counterpart.

We develop a stylized model that leverages the data from Oportunidades to analyze whether schooling exerts a non-monotonic effect on the decision to migrate to the US. We use our incentive framework to identify the causal effects of educational attainment on migration. Our findings provide strong evidence that completing middle school increases the likelihood of migration, while further education beyond middle school does not. We estimate the model using novel machine learning techniques developed by Navjeevan, Pinto, and Santos (2023), which offer superior flexibility and predictive accuracy compared to traditional two-stage least squares (2SLS) approaches.

This paper adds to the growing literature on using revealed preference analysis to aid the identification of causal parameters in policy evaluations. Recent examples of this growing literature include Kline and Walters (2016), Kline and Tartari (2016), Pinto (2022), Feller et al. (2016), Kamat (2021), Mountjoy (2021), and Brinch et al. (2017).

Our empirical analysis contributes to the substantial literature evaluating the Oportunidades program. Our findings support the hypothesis of a non-monotonic relationship between the migration of impoverished Mexicans to the US and their education levels. Additionally, this paper joins a growing body of research applying innovative machine learning techniques (Chernozhukov et al., 2022; Smucler et al., 2019) to perform policy evaluations.

This paper is organized as follows. Section 2 describes our notation and presents a general criterium for identifying causal parameters in IV models. Section 3 introduces our revealed preference framework and demonstrates how it relates to several works in the literature. Section 4 investigates how patterns of choice incentives yield identification conditions in IV models. Section 5 presents our empirical application. Section 6 concludes.

2 Setup and Notation

We consider a IV model consisting of three observed variables: a categorical instrument Z that takes N_Z values in $\mathcal{Z} = \{z_1, ..., z_{N_Z}\}$; a multiple treatment choice T that takes N_T values in $\mathcal{T} = \{t_1, ..., t_{N_T}\}$; and an outcome $Y \in \mathbb{R}$. Let $Y(z,t)$ be the counterfactual outcome Y when (Z,T) are fixed to $(z,t) \in \mathcal{Z} \times \mathcal{T}$, and $Y(t)$ be the counterfactual outcome when T is fixed to $t \in \mathcal{T}$.[3] Let $D_t = \mathbf{1}[T = t]; t \in \mathcal{T}$ be the binary indicator for the treatment choice $t \in \mathcal{T}$ and $D_z = \mathbf{1}[Z = z]; z \in \mathcal{Z}$ be the indicator for the IV-value $z \in \mathcal{Z}$.

The *core properties* of the IV model are:

$$\textbf{Exclusion Restriction :} \quad Y(z,t) = Y(t) \text{ for all } (z,t) \in \mathcal{Z} \times \mathcal{T}. \tag{1}$$

$$\textbf{Exogeneity of the IV:} \quad Z \perp\!\!\!\perp (Y(t), T(z)) \text{ for all } (z,t) \in \mathcal{Z} \times \mathcal{T}. \tag{2}$$

$$\textbf{Relevance of the IV:} \quad E\left([D_{z_1}, ..., D_{z_{N_Z}}]'[D_{t_1}, ..., D_{t_{N_T}}]\right) \text{ has full rank.} \tag{3}$$

The exclusion restriction implies that Z affects Y only through its impact on T. The exogeneity assumption means that the instrument Z is as good as randomly assigned. Finally, the relevance assumption states that Z affects T. All variables belong to the probability space $(\mathcal{I}, \mathcal{F}, P)$ where $i \in \mathcal{I}$ denotes an individual. For notational simplicity, we suppress the baseline variables $\boldsymbol{X}$. All analyses can be understood as conditioned on $\boldsymbol{X}$.

The *response vector* $\boldsymbol{S} \equiv [T(z_1), ..., T(z_{N_Z})]'$ is the unobserved N_Z-dimensional vector that stacks the counterfactual choices $T(z)$ across the IV-values z in $\mathcal{Z}$. The elements $\boldsymbol{s}$ in the support of the response vector, $\mathcal{S} = \{\boldsymbol{s}_1, ..., \boldsymbol{s}_{N_S}\}$, are called the *response types*, or simply *types*. The *response matrix* $\boldsymbol{R} \equiv [\boldsymbol{s}_1, ..., \boldsymbol{s}_{N_S}]$ is the $N_Z \times N_S$ that assembles types as columns and whose rows correspond to IV-values.

To fix ideas, consider the Local Average Treatment Effects (LATE) model of Imbens and

[3]This notation uses the potential outcome framework of Holland (1986); Rubin (1978). For a discussion on causality and the fixing operation, see Heckman and Pinto (2013, 2022).

Angrist (1994). The model employs a binary instrument $Z \in \{z_0, z_1\}$ and a binary treatment $T \in \{t_0, t_1\}$. The response vector $\boldsymbol{S} = [T(z_0), T(z_1)]'$ admits four possible types: never-takers $\boldsymbol{s}_{\text{nt}} = [t_0, t_0]'$, compliers $\boldsymbol{s}_{\text{c}} = [t_0, t_1]'$, always-takers $\boldsymbol{s}_{\text{at}} = [t_1, t_1]'$, and defiers $\boldsymbol{s}_{\text{d}} = [t_1, t_0]'$. Without any additional assumption, its response matrix is given by:

$$\boldsymbol{R} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_{c} & \boldsymbol{s}_{at} & \boldsymbol{s}_{d} \end{array} \\ \begin{bmatrix} t_0 & t_0 & t_1 & t_1 \\ t_0 & t_1 & t_1 & t_0 \end{bmatrix} \end{array} \begin{array}{c} \\ T(z_0) \\ T(z_1) \end{array} \tag{4}$$

Imbens and Angrist (1994) identify this model by assuming a monotonicity condition that eliminates the defiers $\boldsymbol{s}_d$. This identification assumption is part of a broader class of choice restrictions, which we examine in the next section.

The identification of causal parameters in IV models requires expressing the moments of observed data as a function of counterfactual parameters. This relationship is captured by the following equation from Heckman and Pinto (2018):

$$\underbrace{E(Y|T=t, Z=z)P(T=t|Z=z)}_{\text{Observed}} = \sum_{\boldsymbol{s} \in \mathcal{S}} \underbrace{\mathbf{1}[T=t|\boldsymbol{S}=\boldsymbol{s}, Z=z]}_{\text{Known}} \cdot \underbrace{E(Y(t)|\boldsymbol{S}=\boldsymbol{s})P(\boldsymbol{S}=\boldsymbol{s})}_{\text{Unobserved}} \ \forall (z,t) \in \mathcal{Z} \times \mathcal{T}. \tag{5}$$

The left-hand side of (5) comprises two observed quantities: the conditional expectation $E(Y|T=t, Z=z)$, and propensity score $P(T=t|Z=z)$.[4] The first term of the right-hand side of (5) is deterministic since choice T is known given IV-value z and type $\boldsymbol{s}$. The following term comprises two unobserved quantities: the expected value of counterfactual outcomes conditioned on response types $E(Y(t)|\boldsymbol{S}=\boldsymbol{s})$, and the type probabilities $P(\boldsymbol{S}=\boldsymbol{s})$. We can express (5) using the following matrix representation:

$$\boldsymbol{Q}_Z(t) \odot \boldsymbol{P}_Z(t) = \boldsymbol{B}_t \left(\boldsymbol{Q}_S(t) \odot \boldsymbol{P}_S \right) \quad \text{for all } t \in \mathcal{T}, \tag{6}$$

where $\boldsymbol{Q}_Z(t) \equiv [E(Y|T=t, Z=z_1), ..., E(Y|T=t, Z=z_{N_Z})]'$ is the observed vector of outcome expectations; $\boldsymbol{P}_Z(t) \equiv [P(T=t|Z=z_1), ..., P(T=t|Z=z_{N_Z})]$ is the observed vector of propensity scores; $\boldsymbol{Q}_S(t) \equiv [E(Y(t)|\boldsymbol{S}=\boldsymbol{s}_1), ..., E(Y(t)|\boldsymbol{S}=\boldsymbol{s}_{N_S})]$, is the unobserved vector of counterfactual outcomes; $\boldsymbol{P}_S \equiv [P(\boldsymbol{S}=\boldsymbol{s}_1), ..., P(\boldsymbol{S}=\boldsymbol{s}_{N_S})]$ is the unobserved vector of type probabilities; and $\odot$ denotes element-wise (Hadamard) multiplication. Finally, $\boldsymbol{B}_t \equiv \mathbf{1}[\boldsymbol{R}=t]$ is the $N_Z \times N_S$ binary matrix that takes value one if the entry in $\boldsymbol{R}$ is t and zero otherwise. The matrices $\boldsymbol{B}_t$ typically have full row rank since the number of types N_S far exceeds the number of IV-values N_Z. We use $\boldsymbol{B}_t[\cdot, \boldsymbol{s}]$ and $\boldsymbol{B}_t[z, \cdot]$ for the $\boldsymbol{s}$-column and z-row of $\boldsymbol{B}_t$ respectively. Under this notation, we can state the following identification criteria:[5]

[4] Equation (5) holds for any real-valued function $g : \mathbb{R} \to \mathbb{R}$ and for $(z,t) \in \mathcal{Z} \times \mathcal{T}$, that is:

$$E(g(Y)|T=t, Z=z)P(T=t|Z=z) = \sum_{\boldsymbol{s} \in \mathcal{S}} \mathbf{1}[T=t|\boldsymbol{S}=\boldsymbol{s}, Z=z] \cdot E(g(Y(t))|\boldsymbol{S}=\boldsymbol{s})P(\boldsymbol{S}=\boldsymbol{s}).$$

Setting $g(Y) = \mathbf{1}[Y \leq y]$ generates an equation for the cumulative distribution function of counterfactual outcomes. Setting $g(Y) = 1$ generates an equation that relates propensity scores and response type probabilities.

[5] The theorem also holds for the outcome transformation $g(Y)$ for any function $g : \mathbb{R} \to \mathbb{R}$.

Theorem T.1. Let $\boldsymbol{R}$ be the response matrix for a choice model in which the IV Assumptions (1)–(3) hold. Let $\tilde{\mathcal{S}} \subset \mathcal{S}$ represent any subset of response types, and let $|\tilde{\mathcal{S}}|$ denote the number of elements in $\tilde{\mathcal{S}}$. For any choice $t \in \mathcal{T}$, if the binary matrix $\boldsymbol{B}_t = \mathbf{1}[\boldsymbol{R} = t]$ has full row rank, the following condition holds:

$$E(Y(t)|\boldsymbol{S} \in \tilde{\mathcal{S}}) \text{ is identified} \quad \Leftrightarrow \quad \frac{\left(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\right)' \left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1} \left(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\right)}{|\tilde{\mathcal{S}}|} = 1.$$

Moreover, if $E(Y(t)|\boldsymbol{S} \in \tilde{\mathcal{S}})$ is identified, the following must hold:

$$P(\boldsymbol{S} \in \tilde{\mathcal{S}}) = \Big(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\Big)' \left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1} \boldsymbol{P}_Z(t),$$

$$E(Y(t)|\boldsymbol{S} \in \tilde{\mathcal{S}})P(\boldsymbol{S} \in \tilde{\mathcal{S}}) = \Big(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\Big)' \left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1} \left(\boldsymbol{Q}_Z(t) \odot \boldsymbol{P}_Z(t)\right).$$

Proof. See Appendix A.1 □

The theorem provides a straightforward criterion for determining whether the counterfactual outcome $Y(t)$ is identified for a specific set of types $\tilde{\mathcal{S}}$ in any IV model with multiple choices and categorical IV. In particular, it implies that for any choice t and any response type $\boldsymbol{s}$, the counterfactual outcome $E(Y(t)|\boldsymbol{S} = \boldsymbol{s})$ is identified if and only if $\boldsymbol{B}_t[\cdot,\boldsymbol{s}]' \left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1} \boldsymbol{B}_t[\cdot,\boldsymbol{s}] = 1$.[6]

A key insight from this criterion is that identification in IV models depends entirely on the properties of the response matrix $\boldsymbol{R}$. More specifically, the identification of causal parameters arises from the appropriate selection of admissible types that constitute the response matrix $\boldsymbol{R}$. Without further assumptions, the total number of types is $N_T^{N_Z}$, which far exceeds the number of known moments $N_T \cdot N_Z$. This mismatch prevents the point identification of the counterfactual outcomes conditioned on the types. Thus, credible criteria are necessary to restrict the number of admissible types, N_S. The following section introduces a framework that leverages choice incentives to define the set of admissible types in IV models.

3 Exploring Choice Incentives to Eliminate Types

A well-known identification assumption in the LATE model (4) is the monotonicity condition of Imbens and Angrist (1994). The condition states that a change in the instrument from z_0 to z_1 induces agents to choose t_1. This condition can be formalized as:

$$\mathbf{1}[T_i(z_0) = t_1] \leq \mathbf{1}[T_i(z_1) = t_1], \; \forall i. \tag{7}$$

This monotonicity condition eliminates the defiers ($\boldsymbol{s}_d$), which secures the identification of the causal effect for the compliers $E(Y(t_1) - Y(t_0)|\boldsymbol{S} = \boldsymbol{s}_c)$.

Extending the monotonicity condition of Imbens and Angrist (1994) to multiple-choice models is not as straightforward as it may appear. While the binary treatment model leads to a single

[6] We illustrate the application of this result to the LATE model in Appendix A.2.

and intuitive monotonicity condition, the case of multiple choice gives rise to a variety of potential conditions. For instance, the monotonicity condition proposed by Angrist and Imbens (1995) and the unordered monotonicity of Heckman and Pinto (2018) differ considerably in multi-choice settings. Nonetheless, both approaches converge to the same original condition of Imbens and Angrist (1994) when the treatment is binary.

We offer a general framework that leverages choice incentives to explore, generate, and justify monotonicity conditions in multiple-choice models. To do so, we introduce some notation to map choice incentives into identification assumptions.

Let the *incentive matrix* $\boldsymbol{L}$ be a $N_Z \times N_T$ dimensional matrix that characterizes the choice incentives induced by the instrument. Each input $\boldsymbol{L}[z,t]$ denotes the relative incentive of the IV-value z (row) towards the choice $t \in \mathcal{T}$ (column). The inequality $\boldsymbol{L}[z,t] \leq \boldsymbol{L}[z',t]$ means that a change in the IV values from z to z' increases the incentives towards the choice t, making the choice more attractive. In the case of LATE, the IV-value z_1 incentivizes choice t_1. The IV-value z_0 serves as a baseline comparison. Thus, LATE incentives can be characterized by the following matrix:

$$\boldsymbol{L} = \begin{array}{cc} & \begin{array}{cc} t_0 & t_1 \end{array} \\ & \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] \begin{array}{c} z_0 \\ z_1 \end{array} \end{array} \tag{8}$$

The first row of the matrix takes values zero since z_0 provides no incentives. The second row indicates that z_1 incentivizes t_1.

We employ standard revealed preference analysis to develop a *choice rule* that transforms the choice incentives, represented by the incentive matrix $\boldsymbol{L}$, into choice restrictions. This rule is given by:[7]

$$\textbf{Choice Rule:} \quad \text{If } T_i(z) = t \text{ and } \boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \leq \boldsymbol{L}[z',t] - \boldsymbol{L}[z,t] \text{ then } T_i(z') \neq t'. \tag{9}$$

The choice rule states that if agent i prefers choice t over t' under z-incentives ($T_i(z) = t$), and if z'-incentives favor t at least as much as t' ($\boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \leq \boldsymbol{L}[z',t] - \boldsymbol{L}[z,t]$), then the agent will not choose t' over t under z' ($T_i(z') \neq t'$). This rule highlights a fundamental principle of rational choice theory, namely, an individual's preferences toward a choice remain consistent unless there are compelling incentives to choose otherwise. The choice rule generates choice constraints that serve to eliminate types. We illustrate this elimination process by revisiting several prominent examples from the IV literature.

Example **E.1**. Our methodology offers an economic justification for the monotonicity condition in the LATE model. The condition arises from applying revealed preference analysis embodied by the choice rule (9) to the choice incentives represented by the incentive matrix $\boldsymbol{L}$ in (8), that is:

$$T_i(z_0) = t_1, \text{ and } \boldsymbol{L}[z_1,t_0] - \boldsymbol{L}[z_0,t_0] = 0 \leq 1 = \boldsymbol{L}[z_1,t_1] - \boldsymbol{L}[z_0,t_1] \text{ thus } T_i(z_1) \neq t_0.$$

[7] See Appendix A.3 for a formal derivation of the choice rule using revealed preference arguments.

In summary, the LATE incentives lead to the choice restriction $T_i(z_0) = t_1 \Rightarrow T_i(z_1) \neq t_0$. This means that if an agent i chooses t_1 under no incentives (z_0), it will not choose t_0 when the incentives for t_1 are present (z_1). This restriction eliminates the defiers and is equivalent to assuming the customary monotonicity condition in (7).

The next example investigates a multiple-choice model.

Example **E.2**. Kline and Walters (2016) examine the Head Start Impact Study using a three-valued choice model, $T \in \{n, c, h\}$, where h stands for Head Start, c for other preschool programs, and n for no preschool (home-care). The instrument takes values on $Z \in \{z_0, z_1\}$, where z_1 indicates an offer to attend a Head Start school and z_0 denotes no offer. The authors assume that the offer to attend Head Start does not induce children to leave Head Start nor instigate the children to switch between c and n. They express these assumptions by the following choice restriction:

$$T_i(z_0) \neq T_i(z_1) \Rightarrow T_i(z_1) = h \, \forall \, i \in \mathcal{I}. \tag{10}$$

This restriction can also be obtained by applying the choice rule to the model's incentives. Specifically, the incentive matrix of this model is given by:

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} n & c & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_0 \\ z_1 \end{array} \tag{11}$$

Applying the choice rule (9) to the model's incentive matrix generates four choice restrictions:

$$\begin{aligned}
T_i(z_0) = h, &\text{ and } \boldsymbol{L}[z_1, c] - \boldsymbol{L}[z_0, c] = 0 \leq 1 = \boldsymbol{L}[z_1, h] - \boldsymbol{L}[z_0, h] \text{ thus } T_i(z_1) \neq c, \\
T_i(z_0) = h, &\text{ and } \boldsymbol{L}[z_1, n] - \boldsymbol{L}[z_0, n] = 0 \leq 1 = \boldsymbol{L}[z_1, h] - \boldsymbol{L}[z_0, h] \text{ thus } T_i(z_1) \neq n, \\
T_i(z_0) = n, &\text{ and } \boldsymbol{L}[z_1, c] - \boldsymbol{L}[z_0, c] = 0 \leq 0 = \boldsymbol{L}[z_1, n] - \boldsymbol{L}[z_0, n] \text{ thus } T_i(z_1) \neq c, \\
T_i(z_0) = c, &\text{ and } \boldsymbol{L}[z_1, n] - \boldsymbol{L}[z_0, n] = 0 \leq 0 = \boldsymbol{L}[z_1, c] - \boldsymbol{L}[z_0, c] \text{ thus } T_i(z_1) \neq n.
\end{aligned}$$

The first and the second choice restrictions are summarized by $T_i(z_0) = h \Rightarrow T_i(z_1) = h$. The third implies $T_i(z_0) = n \Rightarrow T_i(z_1) \neq c$, while the fourth states that $T_i(z_0) = c \Rightarrow T_i(z_1) \neq n$. Altogether, these restrictions are equivalent to the authors' restriction in (10). These restrictions eliminate four of the nine possible types. The remaining response types are displayed in the following response matrix:

$$\boldsymbol{R} = \begin{array}{c} \begin{array}{ccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 \end{array} \\ \begin{bmatrix} n & c & h & n & c \\ n & c & h & h & h \end{bmatrix} \end{array} \begin{array}{c} \\ T(z_0) \\ T(z_1) \end{array}.$$

A potential critique of the framework presented here is that it requires analytical tools that may seem excessive for simpler models such as LATE. In these cases, relying on monotonicity conditions offers a more parsimonious solution. However, the benefits of the revealed preference framework become more evident when applied to more complex choice models. In such instances, the framework often succeeds in eliminating more types than monotonicity conditions. The following examples demonstrate this advantage.

Example **E.3**. Kirkeboen, Leuven, and Mogstad (2016) investigate a choice model featuring three treatment options (t_0, t_1, t_2) and three IV-values (z_0, z_1, z_2). In this model, z_1 incentivizes choice t_1, z_2 incentivizes choice t_2, while z_0 serves as a baseline with no incentives. The response vector is the 3×1 vector $\boldsymbol{S} = [T(z_0), T(z_1), T(z_2)]'$, where each of the three counterfactual choices $(T(z_0), T(z_1), T(z_2))$ can take on any of the three treatment values (t_0, t_1, t_2). This yields a total of 27 possible response types.

The choice incentives in this model justify two natural monotonicity conditions:

$$\mathbf{1}[T_i(z_0) = t_1] \leq \mathbf{1}[T_i(z_1) = t_1], \quad \text{and} \quad \mathbf{1}[T_i(z_0) = t_2] \leq \mathbf{1}[T_i(z_2) = t_2]. \tag{12}$$

The first condition asserts that an IV-shift from z_0 to z_1 induces agents to choose t_1, while the second condition states that a shift from z_0 to z_2 encourages agents to opt for t_2. These conditions eliminate 12 of the 27 types, which is insufficient to ensure the point identification of any counterfactual outcome.[8]

Employing a revealed preference approach allows for further elimination of response types. The corresponding incentive matrix for this model is as follows:

$$\boldsymbol{L} = \begin{array}{c} \\ \\ \end{array} \begin{array}{ccc} t_0 & t_1 & t_2 \\ \end{array}$$

$$\boldsymbol{L} = \begin{array}{cc} \begin{array}{ccc} t_0 & t_1 & t_2 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} z_0 \\ z_1 \\ z_2 \end{array} \end{array}. \tag{13}$$

Matrix $\boldsymbol{L}$ indicates that z_1 (second row) incentivizes t_1 while z_2 (last row) incentivizes t_2. Appendix A.4 applies choice rule (9) to all combinations of two treatment values $t, t' \in \{t_0, t_1, t_2\}$ and two IV-values $z, z' \in \{z_0, z_1, z_2\}$. These analyses yield five choice restrictions displayed below:

$$\begin{array}{lcl} T_i(z_0) = t_0 & \Rightarrow & T_i(z_1) \neq t_2 \text{ and } T_i(z_2) \neq t_1 \\ T_i(z_0) = t_1 & \Rightarrow & T_i(z_1) = t_1 \text{ and } T_i(z_2) \neq t_0 \\ T_i(z_0) = t_2 & \Rightarrow & T_i(z_1) \neq t_0 \text{ and } T_i(z_2) = t_2 \\ T_i(z_1) = t_2 & \Rightarrow & T_i(z_0) = t_2 \text{ and } T_i(z_2) = t_2 \\ T_i(z_2) = t_1 & \Rightarrow & T_i(z_0) = t_1 \text{ and } T_i(z_1) = t_1 \end{array}. \tag{14}$$

These restrictions are intuitive. For instance, the first choice restriction states that if an agent chooses t_0 under z_0 (no incentives), then it will not choose t_2 under z_1, since z_1 does not incentivize t_2. The agent will not choose t_1 under z_2 either since z_2 does not incentivize t_1 either. The five choice restrictions eliminate 19 response types, including the 12 types eliminated by the monotonicity conditions in (12).[9] The eight types that survive this elimination process are displayed in the following response matrix:

[8] The first monotonicity condition eliminates the six types given by $[t_1, t_2, t']$ or $[t_1, t_3, t']$ for $t' \in \{t_0, t_1, t_2\}$. The second monotonicity condition eliminates another six types: $[t_2, t', t_1]$ or $[t_2, t', t_3]$ for $t' \in \{t_0, t_1, t_2\}$. See Appendix A.4 for this analysis.

[9] See Appendix A.4 for the elimination process and additional analyses of this IV model.

$$
\boldsymbol{R} = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 \\ \end{array}
$$

$$
\boldsymbol{R} = \begin{bmatrix} t_0 & t_1 & t_2 & t_0 & t_0 & t_0 & t_1 & t_2 \\ t_0 & t_1 & t_2 & t_1 & t_0 & t_1 & t_1 & t_1 \\ t_0 & t_1 & t_2 & t_2 & t_2 & t_0 & t_2 & t_2 \end{bmatrix} \begin{array}{c} T(z_0) \\ T(z_1) \\ T(z_2) \end{array} . \tag{15}
$$

The first three response types, $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$, correspond to the always-takers. They refer to agents who choose the same treatment (t_0, t_1, t_2, respectively) regardless of the instrumental value. We refer to type $\boldsymbol{s}_4$ as the intended complier. It comprises agents that are most responsive to the IV incentives. They choose t_0 under no incentives, t_1 when assigned to z_1, and t_2 when assigned to z_2. The remaining four types $\boldsymbol{s}_5, ..., \boldsymbol{s}_8$ are called partial compliers since they choose only two of the three possible choices.

Example **E.4**. Pinto (2022) examines the housing experiment called Moving to Opportunity. The model consists of three neighborhood choices t_h, t_m, t_l denoting high-, medium-, and low-poverty neighborhoods, respectively. Families were randomly assigned to one of the three groups. The control group z_c offers no incentives. The Section Eight group z_8 received a housing voucher that incentivized families to choose either medium-poverty (t_m) or low-poverty (t_l) neighborhoods. The Experimental group z_e received a voucher that incentivized families to live in a low-poverty (t_l) neighborhood. These incentives justify three monotonicity conditions:

$$
\begin{aligned}
\mathbf{1}[T_i(z_c) = t_l] &\leq \mathbf{1}[T_i(z_e) = t_l], \\
\mathbf{1}[T_i(z_c) \in \{t_m, t_l\}] &\leq \mathbf{1}[T_i(z_8) \in \{t_m, t_l\}], \\
\mathbf{1}[T_i(z_e) = t_m] &\leq \mathbf{1}[T_i(z_8) = t_m].
\end{aligned}
$$

The first condition states that an IV-shift from z_c to z_e induces t_l since z_c offers no incentives and z_e incentivizes only t_l. The second condition states that a shift from z_c to z_8 promotes choices t_m or t_h, since z_8 incentivizes both t_m and t_l. The last condition means that a change from z_e to z_8 instigates choice t_m since both z_e, z_8 incentivize t_l but only z_8 incentivizes t_m. These conditions eliminate 14 of the 27 possible response types. The remaining 13 types do not secure the point identification of response-type probabilities or counterfactual outcomes. The revealed preference analysis is more effective in eliminating additional response types. The incentive matrix of this model is:[10]

$$
\boldsymbol{L} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{l} \begin{array}{ccc} t_h & t_m & t_l \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_c \\ z_8 \\ z_e \end{array} . \tag{16}
$$

Applying Choice Rule (9) to the incentive matrix in (16) yields the seven choice restrictions listed below:

[10]See Pinto (2022) for the derivations of these restrictions.

$$
\begin{array}{lcl}
T_i(z_c) = t_l & \Rightarrow & T_i(z_e) = t_l \text{ and } T_i(z_8) \neq t_h \\
T_i(z_c) = t_m & \Rightarrow & T_i(z_e) \neq t_h \text{ and } T_i(z_8) \neq t_h \\
T_i(z_e) = t_m & \Rightarrow & T_i(z_c) = t_m \text{ and } T_i(z_8) = t_m \\
T_i(z_e) = t_h & \Rightarrow & T_i(z_c) = t_h \text{ and } T_i(z_8) \neq t_l \\
T_i(z_8) = t_h & \Rightarrow & T_i(z_c) = t_h \text{ and } T_i(z_e) = t_h \\
T_i(z_8) = t_l & \Rightarrow & T_i(z_e) = t_l \\
T_i(z_c) \neq t_h & \Rightarrow & T_i(z_8) = T_i(z_c)
\end{array}
$$

These restrictions eliminate 20 out of the 27 possible response types, including those eliminated by the monotonicity conditions. The seven types that survive the elimination process are displayed in the following response matrix:

$$
\boldsymbol{R} = \begin{array}{c} \begin{matrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 \end{matrix} \\ \begin{bmatrix} t_h & t_m & t_l & t_h & t_h & t_m & t_h \\ t_h & t_m & t_l & t_m & t_l & t_m & t_m \\ t_h & t_m & t_l & t_l & t_l & t_l & t_h \end{bmatrix} \end{array} \begin{array}{l} \\ T(z_c) \\ T(z_8) \\ T(z_e) \end{array} \tag{17}
$$

These response types enable the point identification of all response type probabilities and most of the counterfactual outcomes.

Example **E.5**. Mountjoy (2022) studies the returns to two- and four-year college degrees. The author uses proximity to college as an IV to college enrollment. Let $T \in \{0, 2, 4\}$ represent the number of years of the college degree. We consider a discrete instrument $Z = (Z_2, Z_4) \in \{0,1\} \times \{0,1\}$ where Z_2 and Z_4 indicate the proximity to two-year and four-year colleges, respectively. We use $T(z_2, z_4)$ for the counterfactual choice. The response vector is $\boldsymbol{S} = [T(0,0), T(0,1), T(1,0), T(1,1)]'$ which This yields a total of 81 possible response types. Proximity serves as an incentive for college enrollment, thereby justifying six natural monotonicity conditions:

$$
\begin{array}{ll}
\mathbf{1}[T_i(1, z_4) = 0] \leq \mathbf{1}[T_i(0, z_4) = 0], & \mathbf{1}[T_i(z_2, 1) = 0] \leq \mathbf{1}[T_i(z_2, 0) = 0], \\
\mathbf{1}[T_i(1, z_4) = 2] \geq \mathbf{1}[T_i(0, z_4) = 2], & \mathbf{1}[T_i(z_2, 1) = 2] \leq \mathbf{1}[T_i(z_2, 0) = 2], \\
\mathbf{1}[T_i(1, z_4) = 4] \geq \mathbf{1}[T_i(0, z_4) = 4], & \mathbf{1}[T_i(z_2, 1) = 4] \geq \mathbf{1}[T_i(z_2, 0) = 4].
\end{array} \tag{18}
$$

These monotonicity conditions state that an increase in the proximity to a two-year college induces agents towards choice 2 and away from choices 0 and 4. Conversely, an increase in the proximity to a four-year college induces agents towards choice 4 and away from choices 0 and 2. These monotonicity conditions eliminate 70 out of the 81 possible response types. The revealed preference analysis is capable of eliminating additional types. The incentive matrix of this choice model is given by:

$$
\boldsymbol{L} = \begin{array}{c} \begin{matrix} 0 & 2 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} (z_2, z_4) \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \tag{19}
$$

The choice restrictions generated by applying choice rule (9) to the incentive matrix in (19) are:[11]

[11] See Appendix A.5 for these derivations.

$$
\begin{aligned}
T_i(0,0) = 0 &\quad \Rightarrow \quad T_i(0,1) \neq 2, \text{ and } T_i(1,0) \neq 4 \\
T_i(0,0) = 2 &\quad \Rightarrow \quad T_i(0,1) \neq 0, \text{ and } T_i(1,0) = T_i(1,1) = 2 \\
T_i(0,0) = 4 &\quad \Rightarrow \quad T_i(1,0) \neq 0, \text{ and } T_i(0,1) = T_i(1,1) = 4 \\
T_i(0,1) = 0 &\quad \Rightarrow \quad T_i(0,0) = 0,\ T_i(1,0) \neq 4, \text{ and } T_i(1,1) \neq 4, \\
T_i(0,1) = 4 &\quad \Rightarrow \quad T_i(1,1) \neq 0 \\
T_i(1,0) = 0 &\quad \Rightarrow \quad T_i(0,0) = 0,\ T_i(0,1) \neq 2, \text{ and } T_i(1,1) \neq 2, \\
T_i(1,0) = 2 &\quad \Rightarrow \quad T_i(1,1) \neq 0
\end{aligned}
$$

These choice restrictions eliminate 72 out of the 81 possible response types, including the types eliminated by the monotonicity conditions in (18). The response matrix containing the nine types that survive this elimination process is given by:

$$
\boldsymbol{R} = \begin{array}{c}
\begin{array}{ccccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 4 \\
0 & 0 & 4 & 4 & 4 & 2 & 4 & 4 & 4 \\
0 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 \\
0 & 2 & 4 & 2 & 4 & 2 & 2 & 4 & 4
\end{bmatrix}
\end{array}
\begin{array}{l} \\ T(0,0) \\ T(0,1) \\ T(1,0) \\ T(1,1) \end{array}
\tag{20}
$$

Generality and Limitations

Our framework is broadly applicable to IV models that can be characterized by an incentive matrix. It encompasses a diverse array of instruments designed to enhance the appeal, accessibility, or affordability of treatment options. These instruments may include financial incentives such as monetary rewards, promotional efforts like advertising campaigns, fiscal policies including tax reductions and subsidies, pricing strategies, or geographical proximity.

The incentive matrix exhibits several desirable properties: (1) it allows an IV-value to incentivize more than one treatment choice; (2) the resulting choice restrictions are invariant to permutations of rows and columns; and (3) choice restrictions are also invariant to strictly monotonic transformations of the matrix.[12]

A key requirement of the incentive matrix is that choice incentives must be comparable. For instance, the incentive matrix is not appropriate to represent the incentives in a schooling experiment that seeks to boost academic performance through monetary prizes or academic awards. These incentives are not easily ranked since some students may favor academic awards, while others may prefer monetary prizes.

4 Which Incentives Justify Monotonicity Conditions?

The revealed preference framework delineated in the preceding section provides a comprehensive methodology for converting choice incentives into choice constraints. An intuitive application of this framework is to investigate which specific configurations of choice incentives substantiate the monotonicity conditions frequently invoked in the IV literature.

[12]Note that if the two rows of an incentive matrix are equal, then the corresponding IV-values are distinguishable in terms of choice incentives. In this case, a researcher could combine these IV-values into a single representative value.

We study the economic content of three monotonicity conditions: the well-known condition of Angrist and Imbens (1995), the unordered monotonicity of Heckman and Pinto (2018), and the extensive margin compliers only (EMCO) discussed by Andresen and Huber (2021); Angrist and Imbens (1995); Rose and Shem-Tov (2021). Additionally, we investigate weaker incentive structures that generate monotonicity conditions for a single treatment status in multi-choice models.

All the theorems presented here apply to IV models described by Assumptions (1)–(3) and whose choice incentives are determined by an incentive matrix $\boldsymbol{L}$ that satisfies Choice Rule (9).

4.1 Investigating Ordered Monotonicity

Angrist and Imbens (1995) states that a change in the instrument induces all agents towards the same treatment direction:

$$T_i(z) \leq T_i(z') \, \forall \, i, \text{ or } T_i(z) \geq T_i(z') \, \forall \, i \text{ and any } z, z' \in \mathcal{Z}. \tag{21}$$

A celebrated result of Angrist and Imbens (1995) is that their monotonicity condition grants a causal interpretation to the standard Two-Stage Least Squares (2SLS) estimator. Vytlacil (2006) shows that this monotonicity condition is equivalent to assuming an ordered choice model with random thresholds. Thus, this monotonicity condition is commonly perceived as an intrinsic property of treatment choices that exhibit a natural order, such as years of schooling. This assessment is misleading. The primary feature of the condition is not the ordered nature of treatment choices. Instead, this monotonicity posits a relationship between IV-values and counterfactual choices whereby higher rankings of the z-values can be associated with higher rankings of $T_i(z)$. To clarify, it is helpful to express this condition in terms of sequences of IV-values and treatment choices:

Ordered Monotonicity Condition (OMC): There exists an ordered sequence of treatment status $t_1 < \cdots < t_{N_T}$ in $\mathcal{T}$ and a sequence of IV-values $z_1, \ldots, z_{N_Z}$ in $\mathcal{Z}$ such that $T_i(z_1) \leq \cdots \leq T_i(z_{N_Z})$ holds for each $i \in \mathcal{I}$.

OMC is a slightly more inclusive version of the Angrist and Imbens (1995) condition. The monotonicity holds whenever it is possible to assign values to treatment choice T such that a sequence of IV-values produces an increasing sequence of counterfactual choices across all types. In the binary choice model, OMC is equivalent to the monotonicity condition of LATE. To gain intuition, consider the case where $T \in \{1, 2, 3\}$, $Z \in \{z_1, z_2, z_3\}$ and Ordered Monotonicity $T_i(z_1) \leq T_i(z_2) \leq T_i(z_3)$ holds. The response matrix that contains all the admissible response types of this choice model is:

$$\boldsymbol{R} = \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 & \boldsymbol{s}_{10} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 2 & 3 & 3 & 2 & 3 & 3 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ T(z_1) \\ T(z_2) \\ T(z_3) \end{array} \tag{22}$$

The response matrix contains ten types (columns) and adheres to the OMC because the choices within each type, $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{10}$, weakly increase as we move down from one row to the next. In the general case of N_T choices and N_Z IV-values, the OMC yields a total of $\binom{N_T+N_Z-1}{N_T-1}$ admissible types. A choice model is said to be *saturated* w.r.t. OMC if its response matrix contains all types that adhere to the monotonicity condition. Otherwise stated, it is not possible to add another type without violating the condition. Conversely, a model is considered *unsaturated* w.r.t. OMC if the response matrix contains only a subset of the possible types that adhere to the condition.

An example of an unsaturated response matrix w.r.t. OMC is **E.3**. Note that if we reorder the IV-values of that matrix as (z_1, z_0, z_2) and assign the choice values $t_1 = 1$, $t_0 = 2$, $t_2 = 3$, we obtain the following incentive matrix and its corresponding response matrix:

$$\boldsymbol{L} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} z_1 \\ z_0 \\ z_2 \end{array} \end{array}, \quad \boldsymbol{R} = \begin{array}{cc} \begin{array}{cccccccc} \boldsymbol{s}_2 & \boldsymbol{s}_1 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 \end{array} & \\ \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 2 & 2 & 1 & 3 \\ 1 & 2 & 3 & 3 & 3 & 2 & 3 & 3 \end{bmatrix} & \begin{array}{c} T(z_1) \\ T(z_0) \\ T(z_2) \end{array} \end{array} \tag{23}$$

The treatment values of each type are non-decreasing as we progress from one row to another. This pattern implies that $T_i(z_1) \leq T_i(z_0) \leq T_i(z_2)$ holds for all $i \in \mathcal{I}$, thereby satisfying OMC. The choice model is unsaturated with respect to OMC because it does not encompass all ten possible types. Consequently, the choice restrictions imposed by the incentive matrix are more stringent than those dictated by OMC alone.

We now investigate the pattern of choice incentives that justifies the OMC.

Supermodular Incentives: Incentives are termed *supermodular* if there exists a sequence of IV-values $z_1, ..., z_{N_Z}$ and a sequence of treatment choices $t_1, ...t_{N_T}$ such that:

$$\boldsymbol{L}[z_{k+1}, t_j] - \boldsymbol{L}[z_k, t_j] \leq \boldsymbol{L}[z_{k+1}, t_{j+1}] - \boldsymbol{L}[z_k, t_{j+1}], \text{ for any } j = 1, ..., N_Z - 1, \text{ and } k = 1, ..., N_Z - 1. \tag{24}$$

The choice incentives $\boldsymbol{L}$ are supermodular if the difference in incentives across IV-values weakly increases as we transition to higher-ranked treatment choices. This pattern includes choice incentives that progressively increase in response to higher values of both IV-values and treatment statuses. We term an incentive matrix $\boldsymbol{L}$ *strictly supermodular* if all the inequalities in (24) are strictly enforced. We are now equipped to state the following result:

Theorem T.2. OMC holds if and only if incentives are supermodular. Moreover, a model with strictly supermodular incentives generates a saturated response matrix w.r.t. OMC.

Proof. See Appendix A.7. □

Theorem **T.2** asserts that supermodular incentives ensure the OMC. For notational convenience, let $\Delta\boldsymbol{L}$ be the row-difference of an incentive matrix $\boldsymbol{L}$:

$$\Delta\boldsymbol{L}[k, j] = (\boldsymbol{L}[z_{k+1}, t_j] - \boldsymbol{L}[z_k, t_j]); \;\; k = 1, .., N_Z - 1; \; j = 1, ..., N_T.$$

Incentives are supermodular if $\Delta\boldsymbol{L}$ is weakly increasing in both row and column dimensions. Furthermore, incentives are strictly supermodular the columns of $\Delta\boldsymbol{L}$ are strictly increasing, namely,

$\Delta \boldsymbol{L}[k,j] < \Delta \boldsymbol{L}[k,j+1]$ for $k = 1, ..., N_Z - 1$. Examples of supermodular incentives for $T \in \{1,2,3\}$ and $Z \in \{z_1, z_2, z_3\}$ are:

$$\boldsymbol{L} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array} \Rightarrow \Delta \boldsymbol{L} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 5 \end{bmatrix} \tag{25}$$

$$\boldsymbol{L} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array} \Rightarrow \Delta \boldsymbol{L} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix} \tag{26}$$

$$\boldsymbol{L} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array} \Rightarrow \Delta \boldsymbol{L} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

The first incentive matrix (25) displays a Vandermonde matrix, which exhibits increasing incentives in both row and column dimensions. This matrix satisfies strictly supermodularity, as the columns of $\Delta \boldsymbol{L}$ are strictly increasing. According to **T.2**, these incentives yield the saturated response matrix displayed in (22).

The second incentive matrix (26) illustrates a pattern where z_1 offers full incentives for choice 1, z_3 offers full incentives for choice 3, and z_2 splits incentives between choices 1 and 3. Strictly supermodularity also holds since the columns of $\Delta \boldsymbol{L}$ are strictly increasing. Again, **T.2** ensures that these incentives yield the saturated response matrix in (22).

The final incentive matrix (27) revisits the example in equation (23). This matrix exhibits supermodularity since the columns in $\Delta \boldsymbol{L}$ are weakly increasing. However, it does not fulfill the criteria for strict supermodularity because the columns of $\Delta \boldsymbol{L}$ are not strictly increasing. These incentives generate the unsaturated response matrix in (23), which satisfies OMC and has fewer types than the saturated version. It is noteworthy that having fewer types provides additional identification power because the incentive matrix imposes more choice restrictions compared to the incentives that generate the saturated response matrix.

4.2 Investigating Unordered Monotonicity

Heckman and Pinto (2018) propose an Unordered Monotonicity Condition that applies to treatment choices that are not ordered. The condition states that for each pair of IV-values $(z, z') \in \mathcal{Z}^2$ and for each $t \in \mathcal{T}$,

$$\mathbf{1}[T_i(z) = t] \leq \mathbf{1}[T_i(z') = t] \,\forall\, i \text{ or } \mathbf{1}[T_i(z) = t] \geq \mathbf{1}[T_i(z') = t] \,\forall\, i. \tag{28}$$

The condition means that for each of the choices t, an IV-change must induce all agents either towards t or away from t. Heckman and Pinto (2018) show that this condition naturally arises in a range of IV settings where treatment choices do not have a clear ordering structure. They also demonstrate that unordered monotonicity allows us to express the indicator for choice t as a latent threshold indicator, similar to the result in Vytlacil (2002). Specifically, $\mathbf{1}[T(z)=t] = \mathbf{1}[P_t(z) \geq U_t]$, where $P_t(z) = P(T=t|Z=z)$ is the propensity score, and $U_t \sim Unif[0,1]$ is an unobserved random variable with a uniform distribution in $[0,1]$ that is statistically independent of Z.[13] Pinto (2022) explores this choice representation to evaluate the Moving to Opportunity intervention as mentioned in example **E.4**. Finally, the monotonicity condition can be equivalently stated in terms of IV-sequences:

Unordered Monotonicity Condition (UMC): For each choice t, there exists a sequence of IV-values $(z_1^{(t)}, ..., z_{N_Z}^{(t)})$ in $\mathcal{Z}$ such that $\mathbf{1}[T_i(z_1^{(t)}) = t] \leq ... \leq \mathbf{1}[T_i(z_{N_Z}^{(t)}) = t]$.

UMC posits that for each treatment choice t there is a sequence of the IV-values that induce agents to choose t. These IV sequences can and do differ across choices $t \in \mathcal{T}$. In contrast, OMC employs a *single sequence* of IV-values that induces all agents to choose higher treatment values.[14] In practical terms, UMC means that it is possible to reorder the rows and columns of the response matrix to generate a lower triangular matrix with respect to each choice t. We revisit the example **E.4** to illustrate this property:

$$\boldsymbol{R} = \begin{array}{c} \begin{matrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 \end{matrix} \\ \begin{bmatrix} t_h & t_m & t_l & t_h & t_h & t_m & t_h \\ t_h & t_m & t_l & t_m & t_l & t_m & t_m \\ t_h & t_m & t_l & t_l & t_l & t_l & t_h \end{bmatrix} \end{array} \begin{array}{l} T(z_c) \\ T(z_8) \\ T(z_e) \end{array}, \quad \boldsymbol{R}_h = \begin{array}{c} \begin{matrix} \boldsymbol{s}_1 & \boldsymbol{s}_7 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_6 \end{matrix} \\ \begin{bmatrix} t_h & t_m & t_m & t_l & t_m & t_l & t_m \\ t_h & t_h & t_l & t_l & t_m & t_l & t_l \\ t_h & t_h & t_h & t_h & t_m & t_l & t_m \end{bmatrix} \end{array} \begin{array}{l} T(z_8) \\ T(z_e) \\ T(z_c) \end{array}, \tag{29}$$

$$\boldsymbol{R}_m = \begin{array}{c} \begin{matrix} \boldsymbol{s}_2 & \boldsymbol{s}_6 & \boldsymbol{s}_4 & \boldsymbol{s}_7 & \boldsymbol{s}_1 & \boldsymbol{s}_3 & \boldsymbol{s}_5 \end{matrix} \\ \begin{bmatrix} t_m & t_l & t_l & t_h & t_h & t_l & t_l \\ t_m & t_m & t_h & t_h & t_h & t_l & t_h \\ t_m & t_m & t_m & t_m & t_h & t_l & t_l \end{bmatrix} \end{array} \begin{array}{l} T(z_e) \\ T(z_c) \\ T(z_8) \end{array}, \quad \boldsymbol{R}_l = \begin{array}{c} \begin{matrix} \boldsymbol{s}_3 & \boldsymbol{s}_5 & \boldsymbol{s}_4 & \boldsymbol{s}_6 & \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_7 \end{matrix} \\ \begin{bmatrix} t_l & t_h & t_h & t_m & t_h & t_m & t_h \\ t_l & t_l & t_m & t_m & t_h & t_m & t_m \\ t_l & t_l & t_l & t_l & t_h & t_m & t_h \end{bmatrix} \end{array} \begin{array}{l} T(z_c) \\ T(z_8) \\ T(z_e) \end{array}. \tag{30}$$

The matrix $\boldsymbol{R}$ is the original response matrix of example **E.4**. Matrix $\boldsymbol{R}_h$ rearranges the columns and rows of the original matrix to generate a lower triangular matrix w.r.t. t_h. This ordering reveals that the number of types taking value t_h increase as we move along the IV-sequence z_8, z_e, z_c.[15] This means that the IV-sequence z_8, z_e, z_c induce agents to choose t_h and the following inequality holds:

$$\mathbf{1}[T_i(z_e) = t_h] \leq \mathbf{1}[T_i(z_8) = t_h] \leq \mathbf{1}[T_i(z_c) = t_h] \; \forall \, i \in \mathcal{I}.$$

Matrices $\boldsymbol{R}_m$ and $\boldsymbol{R}_l$ show that it is also possible to generate lower triangular matrices w.r.t. t_m and t_l via row and column permutations. Consequently, UMC is satisfied. In contrast, OMC

[13] Heckman and Pinto (2018) assume a general model where choice $T = f(Z, \boldsymbol{V})$ is a function of the instrument Z and an absolutely continuous unobserved random vector $\boldsymbol{V}$ that is statistically independent of Z.

[14] UMC does not imply or is implied by OMC in multiple choice models, but they do collapse to the monotonicity condition of Imbens and Angrist (1994) in the case of a binary choice.

[15] Under z_8, only $\boldsymbol{s}_1$ takes the value t_h. Under z_e, the types $\boldsymbol{s}_1$ and $\boldsymbol{s}_7$ take the value t_h, and under z_c, the types that take the value t_h are $\boldsymbol{s}_1, \boldsymbol{s}_7, \boldsymbol{s}_4, \boldsymbol{s}_5$.

does not hold because it is not possible to assign values to treatment choices t_h, t_m, t_l that ensure increasing sequences of counterfactuals $T_i(z_c) \leq T_i(z_8) \leq T_i(z_e)$ across all types.[16] Finally, the response matrix is said to be *saturated* w.r.t. the UMC because it is not possible to add another response type without violating UMC.

A binary matrix is called *lonesum* if it can be transformed into lower triangular matrix via row and column permutations (Ryser, 1957). Thus we can state that UMC holds of and only if when all the binary matrices $\boldsymbol{B}_t \equiv \mathbf{1}[\boldsymbol{R} = t]; t \in \mathcal{T}$ are lonesum. A simple criterion to verify if a response matrix satisfies the UMC is to check if the matrix does not contain a 2×2 submatrix in which the diagonal contains a choice t and the off-diagonal does not.[17] This prohibit pattern prevents us to transform the response matrix into a lower triangular matrix of t-values as illustrated in equations (29)–(30). For instance, the response matrix (23) satisfies OMC but does not satisfy UMC since 2×2 submatrix formed by columns $\boldsymbol{s}_5, \boldsymbol{s}_6$, and rows z_1, z_3, has choice 2 in its diagonal but lacks choice 2 in its off-diagonal. For instance, the response matrix (23) satisfies OMC but does not satisfy UMC, since the 2×2 submatrix formed by columns $\boldsymbol{s}_5, \boldsymbol{s}_6$, and rows z_1, z_3 displays the prohibit pattern: it has choice 2 on its diagonal but lacks choice 2 on its off-diagonal.

We introduce the concept of *monotonic incentives*, which ensures UMC in binary incentive matrices:

$$\textbf{Monotonic Incentives:} \quad \text{for any } z, z' \in \mathcal{Z}, \quad \boldsymbol{L}[z', t] \leq \boldsymbol{L}[z, t] \forall\, t \in \mathcal{T} \text{ or } \boldsymbol{L}[z', t] \geq \boldsymbol{L}[z, t] \forall\, t \in \mathcal{T}. \tag{31}$$

Monotonic incentives imply the existence of an IV-sequence in which the incentives in $\boldsymbol{L}$ weakly increase for all choices. In the case of 3×3 binary matrices, there are four non-equivalent matrices that display monotonic incentives:[18]

$$\boldsymbol{L} = \begin{array}{c c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array}, \quad \boldsymbol{L} = \begin{array}{c c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array}, \quad \boldsymbol{L} = \begin{array}{c c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array}, \quad \boldsymbol{L} = \begin{array}{c c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \end{array}.$$

These matrices satisfy the monotonic incentive criteria because $\boldsymbol{L}[z_1, t] \leq \boldsymbol{L}[z_2, t] \leq \boldsymbol{L}[z_3, t]$ holds for all $t \in \{t_1, t_2, t_3\}$. They are lonesum matrices since they are binary and triangular. Indeed, if $\boldsymbol{L}$ is binary, the concept of monotonic incentives is equivalent to $\boldsymbol{L}$ being lonesum. In particular, the last incentive matrix above is equivalent to the one in the example **E.4**.[19]

Theorem T.3. UMC holds for all binary incentive matrices $\boldsymbol{L}$ satisfying monotonic incentives.

Proof. See Appendix A.8. □

An alternative way to express the Theorem **T.3** is:

[16]For instance, selecting choice values such that $t_h < t_m < t_l$ results in an increasing sequence of treatment values for response type $\boldsymbol{s}_4 = [t_h, t_m, t_l]'$, but it fails to generate an increasing sequence for the type $\boldsymbol{s}_7 = [t_h, t_m, t_h]'$.

[17]See Heckman and Pinto (2018) for a discussion on these properties.

[18]Here, we only examine incentive matrices that do not contain identical rows.

[19]Two matrices are said to be equivalent if they have the same dimension and one can be transformed into the other via row and column permutations.

Corollary C.1. $\boldsymbol{L}$ is lonesum $\Rightarrow$ $\boldsymbol{B}_t$ is lonesum for all $t \in \mathcal{T}$.

Indeed, if the incentive matrix $\boldsymbol{L}$ is lonesum, then the monotonic incentives condition is satisfied. According to Theorem **T.3**, the choice rule imposes restrictions that result in a response matrix $\boldsymbol{R}$, which satisfies the UMC.Consequently, this implies that all binary matrices $\boldsymbol{B}_t \equiv \mathbf{1}[\boldsymbol{R} = t]$ are lonesum.

Monotonic incentives (31) in a simple criteria that ensures UMC. However, this criterion does not cover all types of incentives that are capable of inducing UMC. To establish a more general criterion, it is necessary to study the incentive patterns that induce monotonicity for a single choice t, namely:

$$\mathbf{1}[T_i(z) = t] \leq \mathbf{1}[T_i(z') = t] \,\forall\, i \text{ or } \mathbf{1}[T_i(z) = t] \geq \mathbf{1}[T_i(z') = t] \,\forall\, i \in \mathcal{I} \text{ and any } z, z' \in \mathcal{Z}. \tag{32}$$

The monotonicity condition above describes an indicator inequality in which a change in the instrument induce all agents towards choice t or away from choice t. The condition focuses on a single choice t. UMC arises when this condition holds for all $t \in \mathcal{T}$. The t-monotonic incentives, described below, is central in generating the monotonicity condition of the choice indicator (32):

t-**Monotonic Incentives:** $\boldsymbol{L}$ is t-monotonic if, for any two IV-values $z, z' \in \mathcal{Z}$, we have that:

$$\boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \leq \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \,\forall\, t' \in \mathcal{T} \text{ or } \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \geq \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \,\forall\, t' \in \mathcal{T}. \tag{33}$$

Incentives are termed t-monotonic if for any instrumental change the incentive difference for the choice t is either the maximum or the minimum incentive difference among all treatment choices.

Theorem T.4. Monotonicity condition (32) holds for choice t if and only if $\boldsymbol{L}$ is t-monotonic.

Proof. See Appendix A.9. □

The theorem states that t-monotonic incentives is a necessary and sufficient condition for the t-monotonicity condition in (32) to hold. A natural consequence of the theorem is:

Corollary C.2. UMC holds if and only if $\boldsymbol{L}$ is t-monotonic for all $t \in \mathcal{T}$.

The corollary follows directly from Theorem **T.4** and the definition of UMC. It asserts that UMC holds if and only if, for any IV-change, the incentive differences for each choice $t \in \mathcal{T}$ are either the maximum or minimum of the differences across all choices. Consequently, the incentive differences $\boldsymbol{L}[z', t] - \boldsymbol{L}[z, t]$ must exhibit at most two distinct values for all treatment values $t \in \mathcal{T}$. The following examples help clarify this property:

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} t_h & t_m & t_l \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_c \\ z_8 \\ z_e \end{array} \;\therefore\; \begin{array}{r|ccc} & t_h & t_m & t_l \\ \hline \boldsymbol{L}[z_8,t]-\boldsymbol{L}[z_c,t] & 0 & 1 & 1 \\ \boldsymbol{L}[z_e,t]-\boldsymbol{L}[z_c,t] & 0 & 1 & 0 \\ \boldsymbol{L}[z_8,t]-\boldsymbol{L}[z_e,t] & 0 & 0 & 1 \end{array} \tag{34}$$

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array} \;\therefore\; \begin{array}{r|ccc} & t_1 & t_2 & t_3 \\ \hline \boldsymbol{L}[z_2,t]-\boldsymbol{L}[z_1,t] & 1 & 0 & 0 \\ \boldsymbol{L}[z_3,t]-\boldsymbol{L}[z_1,t] & 0 & 1 & 1 \\ \boldsymbol{L}[z_3,t]-\boldsymbol{L}[z_2,t] & \text{-1} & 1 & 1 \end{array} \tag{35}$$

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array} \;\therefore\; \begin{array}{r|ccc} & t_1 & t_2 & t_3 \\ \hline \boldsymbol{L}[z_2,t]-\boldsymbol{L}[z_1,t] & 0 & 1 & 1 \\ \boldsymbol{L}[z_3,t]-\boldsymbol{L}[z_1,t] & 1 & 2 & 2 \\ \boldsymbol{L}[z_3,t]-\boldsymbol{L}[z_2,t] & 1 & 1 & 1 \end{array} \tag{36}$$

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_1 \\ z_0 \\ z_2 \end{array} \;\therefore\; \begin{array}{r|ccc} & 1 & 2 & 3 \\ \hline \boldsymbol{L}[z_0,t]-\boldsymbol{L}[z_1,t] & \text{-1} & 0 & 0 \\ \boldsymbol{L}[z_2,t]-\boldsymbol{L}[z_1,t] & \text{-1} & 0 & 1 \\ \boldsymbol{L}[z_2,t]-\boldsymbol{L}[z_0,t] & 0 & 0 & \text{-1} \end{array} \tag{37}$$

Equation (34) investigates the incentive matrix of example **E.4**. The second matrix displays the incentive differences corresponding to IV-changes (rows) across the treatment statuses (columns). These incentive differences take only two values, zero or one. Thus, for each IV-comparison, the incentive difference of any treatment status is either the maximum or the minimum among the possible values. This property imply that the incentive matrix is t-monotonic for each treatment choice and, according to **C.2**, UMC holds. This result was previously assessed by noting that the matrix is a case of monotonic incentives (31).

Equation (35) presents a binary incentive matrix that does not exhibit monotonic incentives: the IV-change from z_2 to z_3 decreases the incentive for choosing t_1 while it increases the incentive for choosing t_2. However, the differences in incentives for each IV-change (row) take only two values across the treatment statuses. Thus, UMC holds.

Equation (36) introduces an incentive matrix that is not binary. The incentive differences associated with each IV-change also take at most two values across the treatment choices, which implies UMC. See Appendix A.10 for further details on these models and their corresponding response matrices.

Equation(37) reexamines the incentive matrix in (23), that induces OMC. The incentive difference $\boldsymbol{L}[z_2,t]-\boldsymbol{L}[z_1,t]$ takes three values across $t \in \{1,2,3\}$. Thus, according to **C.2**, UMC does not hold. This fact is corroborated by examining the response matrix in (23). The matrix displays the prohibited pattern in the 2×2 submatrix composed of types $\boldsymbol{s}_5, \boldsymbol{s}_6$ and rows z_1, z_2. The submatrix contains the value two in its diagonal but does not contain the value in its off-diagonal.

We can further explore the properties of t-monotonic incentives. A simple method to check for t-monotonicity in binary incentive matrices is to split the incentive matrix $\boldsymbol{L}$ into $\boldsymbol{L}_t^0$ and $\boldsymbol{L}_t^1$ such that $\boldsymbol{L}_t^0$ contains the z-rows $\boldsymbol{L}[z,\cdot]$ such that $\boldsymbol{L}[z,t]=0$ and $\boldsymbol{L}_t^1$ contains the z-rows such that $\boldsymbol{L}[z,t]=1$. Under this notation, we can present the following result:

Corollary C.3. For any binary incentive matrix $\boldsymbol{L}$, incentives are t-monotonic for a choice $t \in \mathcal{T}$ if and only if matrices $\boldsymbol{L}_t^1$ and $\boldsymbol{L}_t^0$ are lonesum.

Proof. See Appendix A.11. □

The corollary establishes that t-monotonicity is satisfied in a binary incentive matrix $\boldsymbol{L}$ if and only if the sub-matrices $\boldsymbol{L}_t^1$ and $\boldsymbol{L}_t^0$ are lonesum. We use example **E.5** to illustrate an application of this corollary:

$$\boldsymbol{L} = \begin{array}{cc} \begin{matrix} 0 & 2 & 4 \end{matrix} & (z_2, z_4) \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{matrix} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{matrix} \end{array} \therefore \quad \begin{matrix} \boldsymbol{L}_2^0 = \begin{array}{cc} \begin{matrix} 0 & 2 & 4 \end{matrix} & (z_2, z_4) \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{matrix} (0,0) \\ (0,1) \end{matrix} \end{array} \\ \\ \boldsymbol{L}_2^1 = \begin{array}{cc} \begin{matrix} 0 & 2 & 4 \end{matrix} & (z_2, z_4) \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{matrix} (1,0) \\ (1,1) \end{matrix} \end{array} \end{matrix} , \quad \begin{matrix} \boldsymbol{L}_4^0 = \begin{array}{cc} \begin{matrix} 0 & 2 & 4 \end{matrix} & (z_2, z_4) \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} (0,0) \\ (1,0) \end{matrix} \end{array} \\ \\ \boldsymbol{L}_4^1 = \begin{array}{cc} \begin{matrix} 0 & 2 & 4 \end{matrix} & (z_2, z_4) \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} & \begin{matrix} (0,1) \\ (1,1) \end{matrix} \end{array} \end{matrix} .$$

The first matrix displays the incentive matrix $\boldsymbol{L}$ of example **E.5**. Matrices $\boldsymbol{L}_2^0$ and $\boldsymbol{L}_2^1$ split $\boldsymbol{L}$ according to the incentives of choice 2. These matrices do not contain the prohibit pattern (the 2×2 identity matrix). Thus, according to **C.3**, the incentive matrix $\boldsymbol{L}$ is 2-monotonic. Matrices $\boldsymbol{L}_4^0$ and $\boldsymbol{L}_4^1$ refer to choice 4. These matrices do not present the prohibited pattern either. Therefore, $\boldsymbol{L}$ is also 4-monotonic. Incentives for choice 0, first column of $\boldsymbol{L}$, are all zero. Thus $\boldsymbol{L}_0^0 = \boldsymbol{L}$, and the matrix displays the prohibit pattern in the columns associated with choices 2 and 4, and rows $(0,1)$ and $(1,0)$. Therefore $\boldsymbol{L}$ is not 0-monotonic and UMC does not hold.

Although UMC does not hold, the incentive matrix exhibits t-monotonicity for choices 2 and 4. The monotonicity condition in (33) is satisfied for these two choices and there must exist IV-sequences capable of inducing agents to select choices 2 and 4. This assertion can be verified by reordering the rows and columns of the response matrix in (20) to produce lower triangular matrices with respect to choices 2 and 4:

$$\boldsymbol{R}_2 = \begin{array}{cc} \begin{array}{ccccccccc} \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_2 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_8 & \boldsymbol{s}_1 & \boldsymbol{s}_3 & \boldsymbol{s}_9 \end{array} & \\ \begin{bmatrix} 2 & 4 & 0 & 4 & 4 & 4 & 0 & 4 & 4 \\ 2 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 4 \\ 2 & 2 & 2 & 2 & 4 & 4 & 0 & 4 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 4 \end{bmatrix} & \begin{array}{l} T(0,1) \\ T(0,0) \\ T(1,1) \\ T(1,0) \end{array} \end{array}$$

$$\boldsymbol{R}_4 = \begin{array}{cc} \begin{array}{ccccccccc} \boldsymbol{s}_9 & \boldsymbol{s}_8 & \boldsymbol{s}_3 & \boldsymbol{s}_5 & \boldsymbol{s}_4 & \boldsymbol{s}_7 & \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_6 \end{array} & \\ \begin{bmatrix} 4 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 2 \\ 4 & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 4 & 4 & 4 & 4 & 2 & 2 & 0 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 & 2 \end{bmatrix} & \begin{array}{l} T(0,1) \\ T(0,0) \\ T(1,1) \\ T(0,1) \end{array} \end{array}$$

The response matrices above corroborate the result that the monotonicity condition (33) holds for choices 2 and 4. The same feature does not apply to choice 0 since the 2×2 submatrix of response types $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$ and rows $T(0,1)$ and $T(1,0)$ displays the prohibit pattern. Thereby UMC does not hold. OMC does not hold either since no IV-sequence yields a weakly increasing sequence of treatment choices across all types. Despite these results, we can still use the t-monotonicity conditions to express the choice model by the following structural equations:

The response matrices above confirm that the monotonicity condition in (33) holds for choices 2 and 4. However, this condition does not extend to choice 0, as the 2×2 submatrix corresponding to response types $\boldsymbol{s}_2$ and $\boldsymbol{s}_3$ and rows $T(0,1)$ and $T(1,0)$ exhibits a prohibited pattern. Consequently, UMC is not satisfied. Likewise, OMC is also not fulfilled since no IV-sequence yields a weakly increasing sequence of treatment choices across all types. Despite these findings, the t-monotonicity conditions can still be employed to represent the choice model through the following structural equations:

$$T = \begin{cases} 0 & \text{if } P_2(Z) < U_2 \text{ and } P_4(Z) < U_4, \\ 2 & \text{if } P_2(Z) \geq U_2, \\ 4 & \text{if } P_4(Z) \geq U_4, \end{cases}$$

where $P_t(Z) \equiv P(T=t|Z)$ denotes the propensity scores for $t \in \{2,4\}$ and $U_t \sim Unif[0,1]; t \in \{2,4\}$ stands for unobserved variables that are statistically independent Z. This structural representation arises from the t-monotonicity property of choices 2 and 4, and the fact that choice 0 is the complement of choices 2 and 4. This representation allows us to express the counterfactual outcomes $Y(2), Y(4)$ as functions of propensity scores $P_2(Z)$ and $P_4(Z)$ respectively, while $Y(0)$ is a function of both propensity scores. Additional identification power emerges when assuming functional forms for these counterfactuals or exploring baseline variables to generate variation in propensity scores. In the case of continuous instruments, this structural representation can be used to identify average treatment effects using the framework proposed by Lee and Salanié (2018).

It is helpful to summarize our analytical progress thus far. We have shown that the incentive matrix of example **E.3** generate a choice model satisfying only OMC. The choice incentives of example **E.4** generates a model adhering solely to UMC. Furthermore, the incentives of example **E.5** result in a choice model wherein either UMC or OMC is applicable. Next section explores incentives that lead to choice models where both UMC and OMC hold.

4.3 Recoding Treatment into a Binary Indicator

The empirical analysis of IV models frequently involves the conversion of a multi-valued treatment into a binary variable that indicates exposure to a treatment. A typical example is to recode years of schooling into a dummy variable for college or high school graduation.[20] Angrist and Imbens (1995) argue that recoding the treatment status is problematic since the common 2SLS estimand recovers a weighted average of effects that does not a have a clear causal interpretation. This problem has been recently studied by Andresen and Huber (2021) and Rose and Shem-Tov (2023). A simple example clarifies this issue.

Consider the IV model where $T \in \{0, 2, 4\}$ denotes years of college education. Let $Z \in \{z_0, z_1\}$, be an instrument where z_1 offers increasing incentives to greater years of college education while z_0 is a baseline comparison that offers no choice incentives. The incentive matrix of this model and its corresponding response matrix are given below:

$$\boldsymbol{L} = \begin{array}{c c} \begin{array}{ccc} 0 & 2 & 4 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} & \begin{array}{c} z_0 \\ z_1 \end{array} \end{array} \quad \text{and} \quad \boldsymbol{R} = \begin{array}{c c} \begin{array}{cccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 \end{array} & \\ \begin{bmatrix} 0 & 2 & 4 & 0 & 0 & 2 \\ 0 & 2 & 4 & 2 & 4 & 4 \end{bmatrix} & \begin{array}{c} T(z_0) \\ T(z_1) \end{array} \end{array}$$

The incentive matrix above satisfies strict supermodularity. According to Theorem **T.2**, these incentives yield a saturated response matrix concerning OMC. Thus, the response matrix consists of six types $\boldsymbol{s}_1$ through $\boldsymbol{s}_6$ ensuring that $T_i(z_0) \leq T_i(z_1)$ holds for all $i \in \mathcal{I}$. Types $\boldsymbol{s}_1$ through $\boldsymbol{s}_3$ are always-takers, while $\boldsymbol{s}_4$ through $\boldsymbol{s}_6$ are compliers.

Suppose a researcher intends to evaluate the causal effect of four-year college graduation and thus recodes the treatment T into the binary variable $D = \mathbf{1}[T = 4]$ that indicates whether the agent has completed a four-year college education. The Wald estimand of the 2SLS regression recovers the following causal response:

$$\frac{E(Y|Z=z_1) - E(Y|Z=z_0)}{E(D|Z=z_1) - E(D|Z=z_0)} = \underbrace{\frac{E(Y(4) - Y(0)|\boldsymbol{s}_5)P(\boldsymbol{s}_5) + E(Y(4) - Y(2)|\boldsymbol{s}_6)P(\boldsymbol{s}_6)}{P(\boldsymbol{S} \in \{\boldsymbol{s}_5, \boldsymbol{s}_6\})}}_{\text{Intended Effect (extra-margin)}} + \underbrace{\frac{E(Y(2) - Y(0)|\boldsymbol{s}_4)P(\boldsymbol{s}_4)}{P(\boldsymbol{S} \in \{\boldsymbol{s}_5, \boldsymbol{s}_6\})}}_{\text{Unintended Effect (intra-margin)}}$$

This estimand comprises an intended effect and an unintended one. The *intended effect* is the weighted average of the causal effect of four-year college graduation against no college, namely, $E(Y(4) - Y(0)|\boldsymbol{s}_5)$, and the effect of four-year versus two-year college, $E(Y(4) - Y(2)|\boldsymbol{s}_6)$. These effects refer to types $\boldsymbol{s}_5$ and $\boldsymbol{s}_6$ which display *extra-margin variation*: T shifts from 0 or 2 to 4 when D changes from zero to one. The *unintended effect* evaluates the causal effect of two-year

[20] Numerous empirical studies undertake a binary conversion of a multi-valued treatment, including Aizer and Doyle (2015); Arteaga (2021); Bhuller et al. (2020); Black et al. (2005); Carneiro et al. (2011); Finkelstein et al. (2012); Kane and Rouse (1995); Mogstad and Wiswall (2016).

college relative to no college, $E(Y(2) - Y(0)|\boldsymbol{s}_4)$. This effect refers to type $\boldsymbol{s}_4$, which displays an *intra-margin variation*: T changes from 0 to 2 while the binary indicator D remains constant.[21] A solution to this problem is to prevent intra-margin treatment variation by eliminating type $\boldsymbol{s}_4$ from the response matrix.

Rose and Shem-Tov (2021) coined the term Extensive Margin Compliers Only (EMCO) for a monotonicity condition that prevents intra-margin treatment variation in IV models with a binary instrument $Z \in \{z_0, z_1\}$ and ordered treatment $T \in \{0, 1, ..., N_T\}$, namely:

$$T_i(z_0) \leq T_i(z_1) \forall\, i \text{ and } T_i(z_1) > T_i(z_0) \Rightarrow T_i(z_0) = 0\, \forall i.$$

Their condition combines a monotonicity condition with a choice restriction to ensure that individuals can only switch from no treatment, $T_i(z_0) = 0$, to some treatment $T_i(z_1) \neq 0$. This restriction clearly prevents the problem of intra-margin variation. We build on their work to devise a more general condition that prevents intra-margin variation and allows for unordered treatment and categorical instruments. For a given treatment status $t \in \mathcal{T}$ and any for any $z, z' \in \mathcal{Z}$, let the t-EMCO condition be defined as:

$$\mathbf{1}[T_i(z) = t] \leq \mathbf{1}[T_i(z') = t] \forall\, i \text{ and } \mathbf{1}[T_i(z) = t'] \geq \mathbf{1}[T_i(z') = t']; \forall\, i \in \mathcal{I}, t' \in \mathcal{T} \setminus \{t\}, \tag{38}$$

$$\text{or } \mathbf{1}[T_i(z) = t] \leq \mathbf{1}[T_i(z') = t] \forall\, i \text{ and } \mathbf{1}[T_i(z) = t'] \geq \mathbf{1}[T_i(z') = t']; \forall\, i \in \mathcal{I}, t' \in \mathcal{T} \setminus \{t\}. \tag{39}$$

The condition means that a change in the instrument induces agents to switch their decisions towards choice t and away from any other choice. It ensures that any treatment switch must involves the choice t, that is, if $T_i(z) \neq T_i(z')$ then it must be the case that $T_i(z) = t$ or $T_i(z') = t$. The condition is also equivalent to stating that there is an IV-sequence $z_1, ..., z_{N_Z}$ such that:

$$\mathbf{1}[T_i(z_k) = t] \leq \mathbf{1}[T_i(z_{k+1}) = t]\, \forall\, i, \tag{40}$$

$$\text{and } \mathbf{1}[T_i(z_k) = t'] \geq \mathbf{1}[T_i(z_{k+1}) = t']\, \forall\, i, \text{ and } \forall\, t' \in \mathcal{T} \setminus \{t\}. \tag{41}$$

The t-EMCO is a particular case of UMC in which the IV-sequence that induces individuals to select treatment t also prevents agents from switching to any of the other choices $t' \in \mathcal{T} \setminus t$. The t-EMCO condition addresses the issue of intra-margin variation by ensuring that all changes in treatment choice must involve t. Specifically, if $T_i(z_k) \neq T_i(z_{k'})$, then either $T_i(z_k) = t$ or $T_i(z_{k'}) = t$. The equation below displays the saturated response matrix for $T \in \{0, 2, 4\}$ and $Z \in \{z_1, z_2, z_3\}$ when t-EMCO holds for $t = 4$:

$$\boldsymbol{R} = \begin{array}{c} \begin{matrix} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 4 & 0 & 2 & 0 & 2 \\ 0 & 2 & 4 & \boxed{4} & \boxed{4} & 0 & 2 \\ 0 & 2 & 4 & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \end{bmatrix} \end{array} \begin{array}{l} \\ T(z_1) \\ T(z_2) \\ T(z_3) \end{array} \tag{42}$$

Response types $\boldsymbol{s}_1$ through $\boldsymbol{s}_3$ are the always-takes. The remaining types, $\boldsymbol{s}_4$ through $\boldsymbol{s}_7$ are

[21] Another drawback highlighted by Andresen and Huber (2021) is that the binary treatment violates the IV exclusion restriction since the IV affects the counterfactual outcomes through channels beyond D.

the compliers. UNC holds since:

$$\begin{aligned}\mathbf{1}[T_i(z_1)=4] &\leq \mathbf{1}[T_i(z_2)=4] \leq \mathbf{1}[T_i(z_3)=4],\\ \text{and } \mathbf{1}[T_i(z_1)=t'] &\geq \mathbf{1}[T_i(z_2)=t'] \geq \mathbf{1}[T_i(z_3)=4] \text{ for } t' \in \{0,1\}.\end{aligned}$$

Compliers do not display intra-margin variation since all choice changes include 4 : choices are 4 or 0 in $\boldsymbol{s}_4, \boldsymbol{s}_6$, and 4 or 2 in $\boldsymbol{s}_5, \boldsymbol{s}_7$. This feature ensures that the Wald estimand evaluates a weighted average of only the intended treatment effect of completing four-year college graduation ($T=4$) versus not ($T \neq 4$).

The t-EMCO is also a special case of OMC since the inequality $T_i(z_k) \leq T_i(z_{k+1})$ in (40)–(41) is satisfied for any assignment of treatment values where t is the minimum value among treatment choices. It is easy to see that a response matrix that is saturated w.r.t. EMCO, is also saturated w.r.t. UMC, but is unsaturated w.r.t. OMC. The theoretical implications associated with both OMC and UMC apply. For instance, the 2SLS estimand computes a weighted average of per-unit treatment effects (Angrist and Imbens, 1995), and each choice can be described by a separable equation on the propensity score and a latent variable (Heckman and Pinto, 2018).

An incentive matrix $\boldsymbol{L}$ is said to have Constant Incentive Gaps w.r.t. a choice t (t-CIG) if for any $z, z' \in \mathcal{Z}$ we have that:

$$t\textbf{-CIG}: \;\; \boldsymbol{L}[z,t'] - \boldsymbol{L}[z',t'] = \boldsymbol{L}[z,t''] - \boldsymbol{L}[z',t''] \,\forall\, t',t'' \in \mathcal{T} \setminus \{t\}. \tag{43}$$

The t-CIG condition states that the incentive difference between two IV-values for all choices other than t is constant. It is equivalent to assuming that the incentives are separable in Z and T, namely, $L[z,t'] = f(z) + g(t')$ for $t' \in \mathcal{T} \setminus \{t\}$ where $f(\cdot)$ and $g(\cdot)$ are any real-valued functions.[22] Two examples of t-CIG incentives for $t=4$ that apply to our college decision model are:

$$\boldsymbol{L} = \begin{array}{c c} \begin{matrix} 0 & 2 & 4 \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} & \begin{matrix} z_3 \\ z_2 \\ z_1 \end{matrix} \end{array}, \text{ and } \;\; \boldsymbol{L} = \begin{array}{c c} \begin{matrix} 0 & 2 & 4 \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} & \begin{matrix} z_3 \\ z_2 \\ z_1 \end{matrix} \end{array}. \tag{44}$$

The first incentive matrix offers increasing incentives to choose 4-year college and no incentives for the remaining choices. The matrix satisfies t-CIG condition for $t=4$ because $\boldsymbol{L}[z,t'] - \boldsymbol{L}[z',t'] = 0$ holds for $t' \in \{0,2\}$ and any $z \neq z'$. The second incentive matrix displays increasing incentives for choices 0 and 2. It satisfies the 4-CIG condition because $\boldsymbol{L}[z_2,t'] - \boldsymbol{L}[z_1,t'] = 1$ for all $t' \in \{0,2\}$ and $\boldsymbol{L}[z_3,t'] - \boldsymbol{L}[z_2,t'] = 2$ for all $t' \in \{0,2\}$. Either of these incentive matrices yields the response matrix in (42). The following theorem provides a general result connecting t-CIG and t-EMCO.

Theorem T.5. t-EMCO holds if and only if t-CIG is satisfied.

Proof. See Appendix A.12. □

[22] It is easy to see that t-CIG incentives satisfy supermodularity (24), but not in a strictly manner. Also, t-CIG incentives satisfy the t-Monotonic condition (33) for all $t \in calT$.

The theorem states that the t-CIG incentives are a necessary and sufficient condition to generate a response matrix where t-EMCO holds. This means that applying the choice rule (9) to t-CIG incentives, such as in (44), generates choice restrictions that yield a response matrix that satisfies the t-EMCO condition.

Corollary C.4. Let $\boldsymbol{L}$ be an incentive matrix where t-CIG (43) holds. The generated response matrix is saturated w.r.t. t-EMCO (38) if and only if $\boldsymbol{L}[z,t] - \boldsymbol{L}[z',t] \neq \boldsymbol{L}[z,t'] - \boldsymbol{L}[z',t']$ for $t' \neq t$ and all $z, z' \in \mathcal{Z}$ such that $z \neq z'$.

Proof. See Appendix A.13. □

The corollary examines incentives that satisfies the t-CIG condition. This means that incentive differences for any IV-change remain constant across all choices other than t. The corollary states that if the incentive difference for choice t differs from that for the other choices, the generated response matrix is saturated w.r.t. t-EMCO. The t-CIG incentive matrices in (44) satisfy this condition. Therefore, these incentive matrices produce the saturated response matrix displayed in (42).

5 An Empirical Exercise

We use the incentives framework outlined in the previous section to examine the impact of human capital on the illegal migration of poor Mexican families to the US.

The US hosts approximately 12 million undocumented residents, with nearly half of whom originating from Mexico. According to Borjas (1987), migration decisions are positively influenced by wage differentials between native and foreign countries, though these benefits are mitigated by the costs associated with migration. His research suggests a negative selection in migration patterns, wherein lower-skilled workers disproportionately gain from relocating to the U.S. This finding is further substantiated by Angelucci (2015), who demonstrates that Oportunidades, Mexico's foremost anti-poverty initiative, has stimulated the emigration of lower-skilled, undocumented migrants to the US.

Behrman et al. (2005) emphasize that the impact of Oportunidades on school attendance enhances basic English proficiency and analytical skills, both of which are pivotal for success in the US labor market. Their analysis suggests a non-monotonic relationship between education and migration: acquiring fundamental skills increases the propensity to migrate, while further accumulation of human capital reduces this likelihood by making the domestic labor market in Mexico more attractive than the US labor market. This pattern is also supported by Chiquiar and Hanson (2005) and Hanson (2006).

The Mexican education system is structured into three fundamental stages: primaria (primary education), encompassing grades 1 through 6; secundaria (junior high school), covering grades 7 through 9; and preparatoria (high school), consisting of grades 10 through 12. Table 1 lists the main skills taught at each education level. Basic English lessons are introduced at secundaria.

Following Behrman et al. (2005), we posit a non-monotonic relationship between educational attainment and migration. Specifically, completing secundaria is hypothesized to positively influence the propensity to migrate, while advancement from secundaria to preparatoria is expected to have a negative impact on migration.

We analyze a decade of data from the Oportunidades Program to evaluate the impact of educational attainment on migration patterns. Oportunidades is a pioneering initiative in Mexico that employs conditional cash transfers to enhance schooling attainment. The program was commenced in 1997 and targeted impoverished Mexican families living in poor rural areas. It randomly assigned 505 villages to either a treatment group (320 villages) or a control group (185 villages). Families in the treated villages received bi-monthly cash transfers, which often amounted to 20% to 30% of their household income. The transfer was contingent upon their school-age children attending school. Households in control villages had to wait for two years before receiving these benefits. For a detailed description of the program, see (Gertler, 2004).

Our study utilizes panel data from 1997 to 2007. Schooling data were collected in 2003, and census data from 1997, 2003, and 2007 were employed to analyze migration patterns. The sample includes over 3,000 individuals living in impoverished rural areas of Mexico. We assess the impact of Oportunidades on U.S. migration among individuals aged 12 to 13 at the program's onset. This cohort is the most affected by the differing schooling incentives between the treatment and control groups.[23] Approximately 18% of males and 10% of females of our sample migrate. Most of them move to the US between the ages of 16 and 22. Table 5 presents a statistical description of baseline variables by gender. As anticipated, baseline variables exhibit a balanced distribution across randomization arms, and none of the differences in means between the treatment groups are statistically significant.

Traditional Evaluation

We denote the randomization arms as $Z \in \{z_0, z_1\}$, where z_1 represents the treated group and z_0 is the control group. Our primary outcome is the migration indicator $Y \in \{0, 1\}$. Given that the principal aim of the Oportunidades Program is to improve educational attainment, a natural modeling approach is to define the treatment T in terms of years of schooling and to assume the OMC, specifically, $T_i(z_0) \leq T_i(z_1)$ for all $i \in \mathcal{I}$. The OMC enables us to use 2SLS regressions to evaluate the causal effect of the schooling treatment T on the migration outcome Y. Table **??** displays the estimates from this widely used method.

[23]Two criteria define the age range: (1) the lower boundary is set high enough to ensure that the schooling survey in 2003 measures the final schooling attainment and (2) The upper boundary is set low enough to include individuals who were 22 years old in 2007 when the migration data was collected.

Table 1: Skills taught by School Level
Biggest expected effect on migration moving from primary to secondary

Primaria (Grades 1-6)	Secundaria (grades 7-9)	Preparatoria (grades 10-12)
Basic Reading and writing skills	Intermediate Reading and writing skills	Advanced reading and writing skills
Basic mathematical skills	Intermediate mathematical skills	Advanced mathematical skills
To search for information	Work-oriented skills in workshops	Specialized technical-level skills for work
	(carpentry, plumbing and electricity, cooking, etc)	Critical thinking
To follow instructions	**Essential skills for communication in English**	Vocational guidance for jobs in Mexico
Basic comprehension of natural	Civic and social participation	Social skills
and social environments	Self-learning and Reproductive health	Time management

Table 2: Statistical Description of Baseline Variables

	Males			Females		
	Treated Mean	Control Mean	Diff. Means	Treated Mean	Control Mean	Diff. Means
Age at Onset	11.930	11.929	0.000	11.872	11.909	-0.036
s.e.	0.572	0.553	0.028	0.567	0.584	0.029
Family Speaks Indigenous Language	0.417	0.469	-0.052	0.409	0.425	-0.016
s.e.	0.493	0.499	0.025	0.492	0.495	0.025
Household Assets Index	624.27	616.39	7.880	618.81	625.23	-6.428
s.e.	88.546	98.092	4.685	89.707	91.044	4.542
Number of Household Members	7.581	7.476	0.105	7.625	7.594	0.031
s.e.	2.195	2.097	0.106	2.118	2.049	0.104
Household Members Younger than 17	4.739	4.702	0.037	4.789	4.767	0.022
s.e.	1.784	1.736	0.087	1.705	1.702	0.085
County USA Migration Index	-0.155	-0.170	0.015	-0.115	-0.183	0.068
s.e.	0.843	0.919	0.045	0.878	0.908	0.046
Home Ownership	0.969	0.951	0.018	0.971	0.955	0.016
s.e.	0.174	0.216	0.010	0.167	0.207	0.010
Schooling at Onset	4.546	4.407	0.140	4.535	4.611	-0.076
s.e.	1.618	1.630	0.081	1.577	1.596	0.080
Sample Size	1027	674		1048	645	

This columns of this table presents the statistical description of baseline variables by gender. The first row associated to each variable displays the treatment mean, control mean and the mean difference for males and females. The second row displays the standard deviation of the treated and control means and the standard error for the difference-in-means estimator.

The table presents estimates from four models for males and females. The models differ in the set of baseline variables they control for.[24] Panel A presents reduced-form estimates of the effect of opportunities on migration. The intervention consistently raised migration rates by approximately three percentage points for males. This result corresponds to a 22% increase in migration likelihood compared to control males. Panel B presents the first-stage regressions. We find that Oportunidades led to an increase of roughly one-fourth of a school year for males.

Panel C displays the 2SLS estimates where the random assignment of Oportunidades acts as an instrumental variable (IV) to evaluate the impact of education on migration. The estimated coefficient for males is around 0.060 and is statistically significant at the 10% significance level. Panel D displays the OLS estimates of schooling on migration, which are not statistically significant.

[24] Model 1 does not include conditioning variables. Model 2 includes age at onset and county migration index. Model 3 adds family characteristics: family members speaking an indigenous language, number of household members, and number of teenagers. Model 4 includes household assets and house ownership.

Table 3: Standard 2SLS Analysis on the Effect of Schooling and Migration

	Males				Females			
	Model 1	**Model 2**	**Model 3**	**Model 4**	**Model 1**	**Model 2**	**Model 3**	**Model 4**
Panel A: Effect of Oportunidades on Migration (Reduced Form)								
Migration	**0.037**	**0.033**	0.029	**0.031**	0.005	-0.002	-0.001	0.001
s.e.	0.020	0.019	0.018	0.018	0.016	0.016	0.015	0.015
p-val	0.061	0.073	0.113	0.094	0.753	0.873	0.937	0.965
Panel B: Effect of Oportunidades on Schooling (First Stage)								
Schooling	**0.502**	**0.522**	**0.533**	**0.521**	0.022	0.044	0.050	0.076
s.e.	0.121	0.122	0.122	0.122	0.126	0.128	0.128	0.128
p-val	0.000	0.000	0.000	0.000	0.860	0.733	0.695	0.554
Panel C: Effects of Schooling on Migration (2SLS estimates)								
2SLS	**0.073**	**0.064**	0.055	0.059	0.227	-0.057	-0.024	0.009
s.e.	0.043	0.039	0.037	0.038	1.478	0.390	0.313	0.205
p-val	0.092	0.099	0.135	0.118	0.878	0.885	0.938	0.965
Panel D: OLS estimates of the impact of Schooling on Migration								
OLS	-0.005	0.000	0.002	0.002	-0.001	-0.001	-0.001	-0.001
s.e.	0.004	0.004	0.004	0.004	0.003	0.003	0.003	0.003
p-val	0.259	0.980	0.663	0.549	0.803	0.719	0.801	0.792

The table comprises four panels. Panel A displays the causal effects of Oportunidades on migration. Panel B displays the effects Oportunidades on Schooling (measured in 2003). Panel C evaluates the causal effect of schooling on migration using the Oportunidades random assignment as an instrument for schooling. Panel D presents the OLS regression of Migration on schooling. Each panel presents the estimates by gender across four models that differ in terms of conditioning variables. Model 1 does not use conditioning variables. Model 2 employs age at onset and county migration index. Model 3 adds family characteristics: family members speak indigenous language, number of household members, and number of teenagers. Model 4 includes household assets and house ownership. Estimates consists on the effect, its standard error and the double-sided p-value associated with inference that tests if the effect is equal to zero. All estimates are based on the standard OLS and 2SLS regressions. Inference employs clustered errors at village levels.

The 2SLS analysis provides a valuable framework for assessing the overall impact of education on migration. Specifically, the 2SLS coefficient estimates a weighted average of the per-unit treatment effect across individuals whose educational attainment increases as the instrument shifts from z_0 to z_1. However, this causal interpretation poses challenges when attempting to elucidate the non-monotonic relationship between education and migration, which is the focus of our investigation. To address this complexity, we propose a stylized model that incorporates the underlying incentives influencing educational choices and the observed patterns in school choices.

5.1 Stylized Model

Figure **??** presents the distribution of educational attainment at the onset of the intervention in 1997 and six years post-intervention in 2003. The data reveal a significant concentration of education levels around the completion of secundaria (9 years of schooling). This schooling stage is critical for our analysis since it is where basic English skills are taught. We explore this fact to construct a stylized model that considers the completion of secundaria as a pivotal milestone in assessing the impact of education on migration.

We define the schooling index $T \in \{1, 2, 3\}$, where $T = 1$ represents education below secundaria (primarily encompassing the completion of primaria), $T = 2$ indicates the completion of secundaria, and $T = 3$ denotes education beyond secundaria.

We proceed to examine the incentives of the Oportunidades program through the lens of our schooling index. Our sample comprises an age cohort of students making decisions about completing secundaria. Students in the treatment group (z_1) received cash transfers throughout their schooling years, thereby providing them with increasing incentives to complete secundaria and pursue further education. Conversely, students in the control group (z_0) received cash transfers only after a few years. These delayed incentives missed the critical period when students typically complete secundaria. Nonetheless, those who did complete secundaria were subsequently incentivized to continue their studies. This incentive scheme is represented by the following incentive matrix:

$$\boldsymbol{L} = \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{array}{c} z_0 \\ z_1 \end{array} \end{array} \tag{45}$$

The incentive matrix provides increasing benefits for the treated participants to continue studying, while offering delayed incentives for control participants to pursue schooling beyond secundaria. Note that the incentive matrix (45) is t-monotonic for all choices. According to **T.4** these incentives must lead to a response matrix that satisfy UMC. Moreover, the incentives are supermodular, but not strictly supermodular. According to **T.2**, these incentives lead to a response matrix is non-saturated w.r.t. OMC.

Applying the Choice Rule (9) to incentives (45) generates the following choice restrictions:

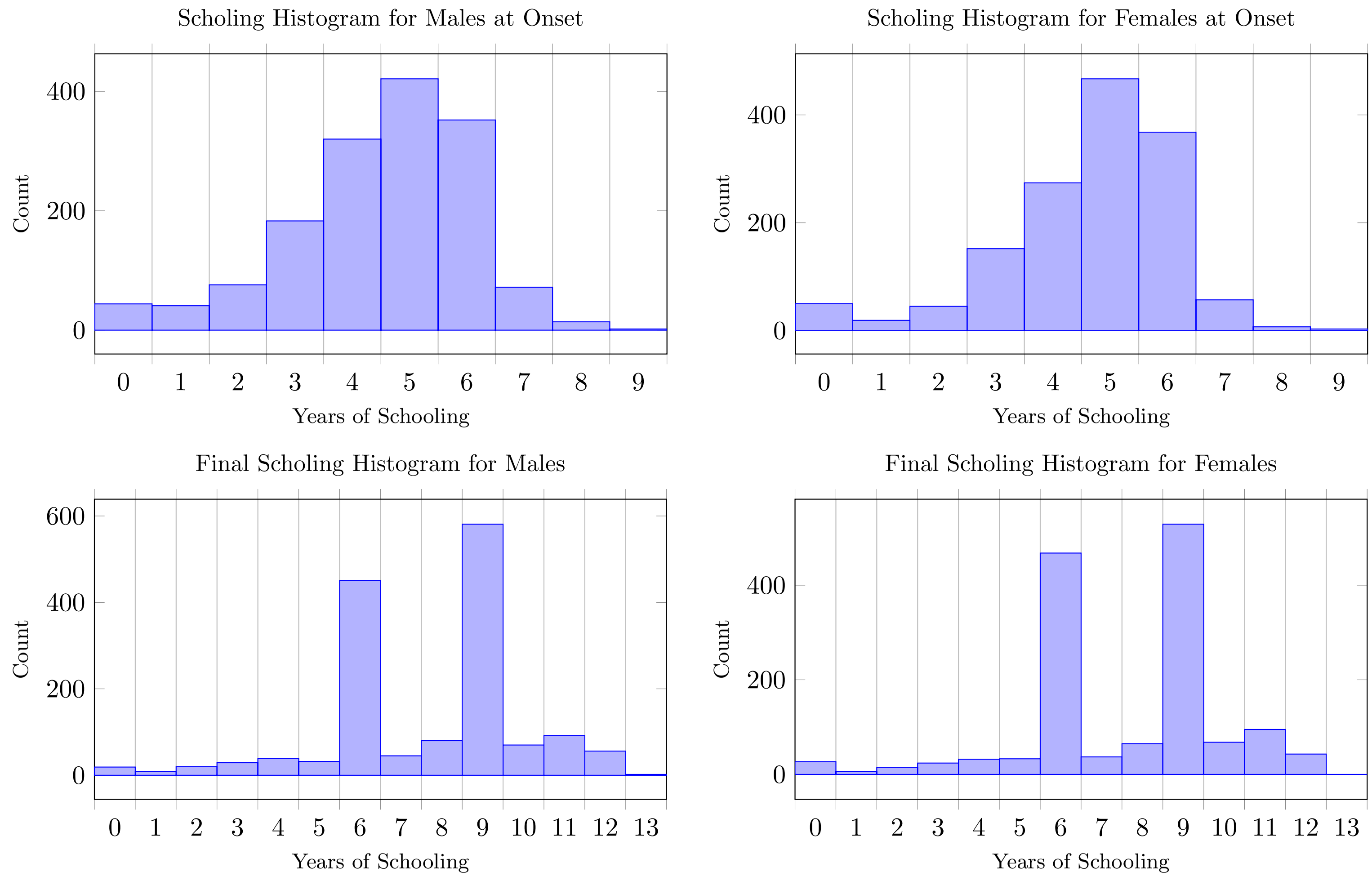
$$\begin{array}{lcl} T_i(z_0) = 2 & \Rightarrow & T_i(z_1) \neq 1 \text{ and } T_i(z_1) \neq 3 \\ T_i(z_0) = 3 & \Rightarrow & T_i(z_1) \neq 1 \text{ and } T_i(z_1) \neq 2 \\ T_i(z_1) = 1 & \Rightarrow & T_i(z_0) \neq 2 \text{ and } T_i(z_0) \neq 3 \\ T_i(z_1) = 2 & \Rightarrow & T_i(z_0) \neq 3 \\ T_i(z_1) = 3 & \Rightarrow & T_i(z_0) \neq 2 \end{array} \tag{46}$$

The resulting response matrix is:

$$\boldsymbol{R} = \begin{array}{c c} \begin{array}{ccccc} \boldsymbol{s}_{11} & \boldsymbol{s}_{22} & \boldsymbol{s}_{33} & \boldsymbol{s}_{12} & \boldsymbol{s}_{13} \end{array} & \\ \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 2 & 3 \end{bmatrix} & \begin{array}{c} T(z_0) \\ T(z_1) \end{array} \end{array}. \tag{47}$$

The response matrix contains three always-takers $\boldsymbol{s}_{11}, \boldsymbol{s}_{22}, \boldsymbol{s}_{33}$, and two compliers $\boldsymbol{s}_{12}, \boldsymbol{s}_{13}$. As expected, the matrix satisfies UMC since the matrix does not display any prohibited pattern. The

Figure 1: Histograms of Years of Schooling at the Onset and After the Intervention



This histogram displays the observed distribution of years of schooling at the onset of the intervention in 1997 and six years later, in 2003. The sample includes participants who were between the ages of 11 and 12 at the beginning of the intervention.

response matrix satisfies OMC since $T_i(z_0) \leq T_i(z_1)$ holds for all agents. The matrix is not saturated w.r.t. OMC since it lacks the type $[2,3]'$.

The identification analysis stems from Theorem T.1. All type probabilities are (just) identified. Always-taker probabilities are given by:[25]

$$\begin{aligned} P(\boldsymbol{s}_{11}|X) &= P(T=1|z_1,X), \\ P(\boldsymbol{s}_{22}|X) &= P(T=2|z_0,X), \\ \text{and } P(\boldsymbol{s}_{33}|X) &= P(T=3|z_0,X). \end{aligned}$$

The probabilities for compliers are identified by:

$$\begin{aligned} P(\boldsymbol{s}_{12}|X) &= P(T=2|z_1,X) - P(T=2|z_0,X), \\ \text{and } P(\boldsymbol{s}_{22}|X) &= P(T=3|z_1,X) - P(T=3|z_0,X). \end{aligned}$$

There are six counterfactual outcomes that are identified. Counterfactual outcomes for always-takers are given by:

$$\begin{aligned} E(Y(1)|\boldsymbol{s}_{11}) &= E(Y|T=1,z_1,X), \\ E(Y(2)|\boldsymbol{s}_{22}) &= E(Y|T=2,z_0,X), \\ \text{and } E(Y(3)|\boldsymbol{s}_{33}) &= E(Y|T=3,z_0,X). \end{aligned}$$

The remaining counterfactuals are identified as LATE-type parameters:

$$\begin{aligned} E(Y(1)|\boldsymbol{S} \in \{\boldsymbol{s}_{12},\boldsymbol{s}_{13}\},X) &= LATE_X(\mathbf{1}[T=1]), \\ E(Y(2)|\boldsymbol{S}=\boldsymbol{s}_{12},X) &= LATE_X(\mathbf{1}[T=2]), \\ E(Y(3)|\boldsymbol{S}=\boldsymbol{s}_{13},X) &= LATE_X(\mathbf{1}[T=3]), \\ \text{where: } LATE_X(W) &\equiv \frac{E(Y\cdot W|Z=z_1,X) - E(Y\cdot W|Z=z_0,X)}{E(W|Z=z_1,X) - E(W|Z=z_0,X)}. \end{aligned}$$

We are most interested in two causal effects: $E(Y(2)-Y(1)|\boldsymbol{S}=\boldsymbol{s}_{12})$, which is the causal effect of completing secundaria on migration, and $E(Y(3)-Y(1)|\boldsymbol{S}=\boldsymbol{s}_{13})$, which is the causal effect of studying beyond secundaria on migration. Unfortunately, these effects are only partially identified since we cannot disentangle $E(Y(1)|\boldsymbol{S}\in\{\boldsymbol{s}_{12},\boldsymbol{s}_{13}\})$ into $E(Y(1)|\boldsymbol{S}=\boldsymbol{s}_{12})$ and $E(Y(1)|\boldsymbol{S}=\boldsymbol{s}_{13})$ without additional assumptions.

A common solution to this problem of partial identification is to invoke the assumption of comparable compliers. Recent examples of this type of assumption include Mountjoy (2022) and Navjeevan et al. (2023). In our case, we have that:

$$\text{Comparable Compliers:} \quad Y(1) \perp\!\!\!\perp \boldsymbol{S}|(T(z_0)\neq T(z_1),X). \tag{48}$$

The assumption states that, conditioned on being a complier and on baseline variables X, the counterfactual outcome $Y(1)$ is independent of the types. It applies only to agents that choose to

[25] We use $P(\boldsymbol{s}|X)$ and $P(T=t|z,X)$ as short-hand notation for $P(\boldsymbol{S}=\boldsymbol{s}|X)$ and $P(T=t|Z=z,X)$ respectively.

not complete secundaria under no incentives (z_0) but would pursue additional schooling if Oportunidades incentives were available (z_1). It implies that the migration outcome for those who choose to not complete secundaria are comparable. Effectively, this assumption establishes the moment equality $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12}, X) = E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13}, X)$, which enables us to solve the problem of partial identification.

5.2 Estimating Type Probabilities

We devise a doubly robust estimator that employs machine learning techniques to evaluate causal parameters. The method stems from the work of Navjeevan, Pinto, and Santos (2023) and it has desirable properties commonly shared by this type of estimator. The method yields asymptotically normal estimators that guarantee double robustness against misspecification (Robins et al., 1995) and possesses the mixed bias property in (Chernozhukov et al., 2018). The method also benefits from a variety of plug-in machine learning techniques, as described in Smucler et al. (2019), Chernozhukov et al. (2022), and Chernozhukov et al. (2022).

To gain intuition, we examine the identification of type probabilities in greater detail. Let $\boldsymbol{P}_{T|X}(t) \equiv [P(T = t|Z = z_0, X), P(T = t|Z = z_1, X)]'$ be the 2×1 vector of choice probabilities across IV-values, and $\boldsymbol{P}_{T|X} \equiv [\boldsymbol{P}_{T|X}(1)', \boldsymbol{P}_{T|X}(2)', \boldsymbol{P}_{T|X}(3)']'$ be the 6×1 vector of propensity scores. Moreover, let the 5×1 vector of type probabilities conditioned on X be:

$$\boldsymbol{P}_{\boldsymbol{S}|X} = [P(\boldsymbol{s}_{11}|X), P(\boldsymbol{s}_{22}|X), P(\boldsymbol{s}_{33}|X), P(\boldsymbol{s}_{12}|X), P(\boldsymbol{s}_{13}|X)]'.$$

These vectors are related by the following equation $\boldsymbol{P}_{T|X} = \boldsymbol{B}\boldsymbol{P}_{\boldsymbol{S}|X}$, where $\boldsymbol{B} \equiv [\boldsymbol{B}_1', \boldsymbol{B}_2', \boldsymbol{B}_3']'$ is the 8×5 binary matrix that stacks the indicator matrices $\boldsymbol{B}_t = \mathbf{1}[\boldsymbol{R} = t]$ across the treatment choices. The response matrix $\boldsymbol{R}$ in (47) enable us to express each of the type probabilities as a linear combination of the propensity scores:

$$P(\boldsymbol{S} = \boldsymbol{s}|X) = \boldsymbol{\nu}_{\boldsymbol{s}}\boldsymbol{P}_{Z|X} \text{ such that } \boldsymbol{\nu}_{\boldsymbol{s}} \equiv \ell_{\boldsymbol{s}}'(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'. \tag{49}$$

The term $\boldsymbol{\nu}_{\boldsymbol{s}}$ is primary in our analysis. It is a known 6×1 vector defined as $\ell_{\boldsymbol{s}}'(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'$, where $\ell_{\boldsymbol{s}}$ is a 5×1 canonic vector that takes value one for type $\boldsymbol{s}$ and zero otherwise. Vector $\boldsymbol{\nu}_{\boldsymbol{s}}$ can be understood as a map $\nu_{\boldsymbol{s}}(z, t)$ from the support of (Z, T) to $\mathbb{R}$. In this notation, we can rewrite equation (49) as:

$$P(\boldsymbol{S} = \boldsymbol{s}|X) = \sum_{t\in\mathcal{T}}\sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(z, t)P(T = t|Z = z, X). \tag{50}$$

To construct the doubly robust estimator, we represent the type probability as the expectation of a function κ such that $P(\boldsymbol{S} = \boldsymbol{s}) = E(\kappa_{\boldsymbol{s}}(T, Z, X))$.[26] The doubly robust estimator is based on the following the orthogonal score representation of type probabilities:

$$P(\boldsymbol{S} \ = \boldsymbol{s}) = \sum_{t\in\mathcal{T}} E_{ZX}\left(\kappa_{\boldsymbol{s}}(t, Z, X)\cdot\left(\mathbf{1}[T = t] - P(T = t|Z, X)\right)\right) + \sum_{t\in\mathcal{T}} E_X\left(\sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(z, t)P(T = t|Z = z, X)\right),$$

[26] See Navjeevan, Pinto, and Santos (2023) for a in-depth discussion of the rationale of this approach.

where $E_{ZX}(\cdot)$ is an expectation over the joint distribution of (Z, X) and $E_X(\cdot)$ is an expectation over X. The identifying moment condition has two nuisance parameters, the function $\kappa_{\boldsymbol{s}}(t, Z, X)$ and the propensity score $P(T = t|Z, X)$. We assess these nuances via plug-in estimators. We evaluate the propensity score $P(T = t|Z, X)$ by $\boldsymbol{h}(Z, X)\boldsymbol{\beta}_t$, and the kappa function $\kappa_{\boldsymbol{s}}(t, Z, X)$ by $\boldsymbol{h}(Z, X)\boldsymbol{\gamma}_{\boldsymbol{s},t}$, where $\boldsymbol{\beta}_t, \boldsymbol{\gamma}_t$ are p-dimensional linear coefficients and $\boldsymbol{h}(Z, X) = [b_1(Z, X), ..., b_p(Z, X)]'$ denotes a p-dimensional vector of function of (Z, X) including all the pairwise interactions of these variables. In our application, $\boldsymbol{h}(Z, X)$ comprises X, Z, and their interaction. Appendix A.14 presents a detailed description of the estimation algorithm.

Table 4 presents the estimated probabilities for each type. The aggregate probability for the always-takers is approximately 0.90, indicating that 90% of the sample comprises students who persist with their schooling choice towards secundaria irrespective of their allocation to either treatment or control groups. The probability for type $\boldsymbol{s}_{11}$ stands at around 0.43, suggesting that nearly half of the sample consists of students who do not complete secundaria, regardless of the incentives from Oportunidades. The probability associated with type $\boldsymbol{s}_{22}$ is close to 0.34, denoting that a third of the sample consistently chooses to finalize their secundaria. Lastly, the probability for type $\boldsymbol{s}_{33}$ is approximately 0.13, implying that a mere 13% of the students opt to pursue education beyond secundaria, irrespective of receiving the Oportunidades incentives or not.

The sum of the probabilities for compliers, $\boldsymbol{s}_{12}$ and $\boldsymbol{s}_{13}$, totals 0.093. This means that about 9% of the students change their choice towards completing secundaria when the incentives provided by Oportunidades are available. The majority of these students, about 7%, consist of participants of type $\boldsymbol{s}_{12}$ who shift from not completing secundaria to completing it. A smaller share of the sample, about 2%, comprises compliers that change from not completing secundaria when assigned to the control to studying beyond secundaria when assigned to the treatment.

5.3 Estimating Causal Effects

We now describe the doubly robust estimator employed to evaluate the counterfactual outcomes. This discussion parallels our earlier examination of type probabilities.

Let $\boldsymbol{E}_{Y|X}(t) \equiv [E(Y \cdot \mathbf{1}[T = t]|Z = z_0, X), E(Y \cdot \mathbf{1}[T = t]|Z = z_1, X)]'$ be the 2×1 vector of conditional outcome moments. As mentioned, $\boldsymbol{B}_t = \mathbf{1}[\boldsymbol{R} = t]$ denotes the binary matrix that indicates which elements in the response matrix $\boldsymbol{R}$ in (47) takes value $t \in \{1, 2, 3\}$. We can then express the five identified counterfactual outcomes – $E(Y(1)|\boldsymbol{s}_{11}, X)$, $E(Y(2)|\boldsymbol{s}_{22}, X)$, $E(Y(3)|\boldsymbol{s}_{33}, X)$ $E(Y(2)|\boldsymbol{s}_{12}, X)$ and $E(Y(3)|\boldsymbol{s}_{13}, X)$ – in the following fashion:

$$E(Y(t)|\boldsymbol{S} = \boldsymbol{s}|X)P(\boldsymbol{S} = \boldsymbol{s}|X) = \boldsymbol{\nu}_{\boldsymbol{s},t}\boldsymbol{E}_{Z|X}(t) \text{ such that } \boldsymbol{\nu}_{\boldsymbol{s},t} \equiv \ell'_{\boldsymbol{s}}\boldsymbol{B}'_t(\boldsymbol{B}_t\boldsymbol{B}'_t)^{-1}, \tag{51}$$

where $\ell_{\boldsymbol{s}}$ is a 5×1 canonic vector that indicates type $\boldsymbol{s}$. Similar to our analysis of type probabilities, we can express $\boldsymbol{\nu}_{\boldsymbol{s},t}$ as a function $\nu_{\boldsymbol{s},t}(z)$ from the support of Z to $\mathbb{R}$. In this notation, we can rewrite

Table 4: Type Probabilities and Causal Effects for Males

Type Probabilities	**Model 1**	**Model 2**	**Model 3**	**Model 4**
$P(\boldsymbol{S} = \boldsymbol{s}_{11})$	**0.436**	**0.436**	**0.435**	**0.436**
s.e.	0.016	0.016	0.016	0.016
p-val	0.000	0.000	0.000	0.000
$P(\boldsymbol{S} = \boldsymbol{s}_{22})$	**0.337**	**0.338**	**0.337**	**0.340**
s.e.	0.019	0.019	0.019	0.019
p-val	0.000	0.000	0.000	0.000
$P(\boldsymbol{S} = \boldsymbol{s}_{33})$	**0.133**	**0.133**	**0.130**	**0.131**
s.e.	0.014	0.014	0.014	0.014
p-val	0.000	0.000	0.000	0.000
$P(\boldsymbol{S} = \boldsymbol{s}_{44})$	**0.071**	**0.071**	**0.070**	**0.068**
s.e.	0.024	0.024	0.025	0.025
p-val	0.004	0.004	0.005	0.007
$P(\boldsymbol{S} = \boldsymbol{s}_{55})$	0.022	0.023	0.026	0.026
s.e.	0.018	0.018	0.018	0.018
p-val	0.206	0.181	0.139	0.149

This table presents the estimates of type probabilities according to the doubly robust orthogonal score estimator described in this section. Estimates are presents for four models that vary in the set of baseline variables X that we seek to condition on. Model 1 does not use baseline variables. Model 2 employs age at onset and county migration index. Model 3 adds family characteristics: family members speak indigenous language, number of household members, and number of teenagers. Model 4 includes household assets and house ownership. Estimates consists on the probability, its standard error and the two-sided p-value associated with inference that tests if the effect is equal to zero. Standard errors are computed using the multiplier bootstrap method.

equation (51) as:

$$E(Y(t)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}]|X)P(\boldsymbol{S} = \boldsymbol{s}|X) = \sum_{z \in \mathcal{Z}} \nu_{\boldsymbol{s},t}(z) E(Y \cdot \mathbf{1}[T = t]|Z = z, X). \tag{52}$$

It is also worth noting that the response types probability associated with the identified outcome counterfactuals can be identified as:

$$P(\boldsymbol{S} = \boldsymbol{s}|X) = \sum_{z \in \mathcal{Z}} \nu_{\boldsymbol{s},t}(z) E(\mathbf{1}[T = t]|Z = z, X). \tag{53}$$

The doubly robust estimator comprises the joint evaluation of the expectation $E(Y(t)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}])$ in (52), the probability $P(\boldsymbol{S} = \boldsymbol{s}|X)$ and then taking the ratio of these estimates. Note that both problems are related since they are associated with the same identification function $\nu_{\boldsymbol{s},t}(z)$. The estimator for $E(Y(t)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}])$ is based on the following orthogonal score:

$$E(Y(t)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}]) = E_{ZX}\Big(Y\kappa_{\boldsymbol{s},t}(Z,X) \cdot \big(Y\mathbf{1}[T = t] - E(Y\mathbf{1}[T = t]|Z, X)\big)\Big) + E_X\left(\sum_{z \in \mathcal{Z}} \nu_{\boldsymbol{s},t}(z) E(Y \cdot \mathbf{1}[T = t]|Z = z, X)\right).$$

The function kappa is such that $E(Y\kappa_{\boldsymbol{s},t}(Z,X)) = E(Y(t)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}])$ and $E(\kappa_{\boldsymbol{s},t}(Z,X)) = P(\boldsymbol{S} =$

$\boldsymbol{s}$). The estimator contains three nuance parameters: the propensity score $P(T = t|Z, X)$ is estimated as by $\boldsymbol{h}(Z, X)\boldsymbol{\beta}_t$, the outcome expectation $E(Y \cdot \mathbf{1}[T = t]|Z = z, X)$ by $\boldsymbol{h}(Z, X)\boldsymbol{\theta}_t$, and the kappa function $\kappa_{\boldsymbol{s},t}(t, Z, X)$ by $\boldsymbol{h}(Z, X)\boldsymbol{\gamma}_{\boldsymbol{s},t}$, where $\boldsymbol{\beta}_t, \boldsymbol{\theta}_t, \boldsymbol{\gamma}_t$ are p-dimensional linear coefficients and $\boldsymbol{h}(Z, X)$ comprises X, Z, and their interaction. The steps of the estimator are closely related to the estimation of type probabilities. Appendix A.15 describes the estimation algorithm in great detail.

Our goal is to evaluate two causal effects: $E(Y(2) - Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12})$ and $E(Y(2) - Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13})$. The procedure outlined in Appendix A.15 is tailored to estimate any counterfactual outcome that is identified according to the response matrix $\boldsymbol{R}$ (47). These include $E(Y(2)|\boldsymbol{S} = \boldsymbol{s}_{12})$ and $E(Y(3)|\boldsymbol{S} = \boldsymbol{s}_{13})$. The procedure could also be used to evaluate $E(Y(1)|\boldsymbol{S} \in \{\boldsymbol{s}_{12}, \boldsymbol{s}_{13}\})$, since it is also identified. The procedure however is not suitable to evaluate $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12})$ and $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13})$ separately.

The additional assumption of comparable compliers (48) enable us to disentangle $E(Y(1)|\boldsymbol{S} \in \{\boldsymbol{s}_{12}, \boldsymbol{s}_{13}\})$ into $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12})$ and $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13})$. The assumption implies that $E(Y(1)|\boldsymbol{S} \in \{\boldsymbol{s}_{12}, \boldsymbol{s}_{13}\}|X) = E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12}|X)$ and $E(Y(1)|\boldsymbol{S} \in \{\boldsymbol{s}_{12}, \boldsymbol{s}_{13}\}|X) = E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13}|X)$. Note however that this assumption does not imply the unconditional equality $E(Y(1)|\boldsymbol{S} \in \{\boldsymbol{s}_{12}, \boldsymbol{s}_{13}\}) = E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12}) = E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13})$ because the distribution of baseline variables X may differ across types $\boldsymbol{s}_{12}$ and $\boldsymbol{s}_{13}$. The modification of the procedure is necessary to account for the difference in the distribution of baseline variables X between types. Navjeevan, Pinto, and Santos (2023) solve the same problem in a different setting involving the mediation analysis of a choice model containing seven types. We adapt their solution to our setting. Appendix A.16 provides a detailed description of the estimation algorithm.

Table 5 presents the causal effects of our model conditioned on different sets of baseline variables. The first panel of the table presents the estimates for $E(Y(2) - Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12})$, which evaluates the casual effect of completing secundaria on migration for the subset of compliers that change from not completing secundaria to completing it when the incentives of Oportunidades are available. These compliers account for about 7% of the sample. We find that completing secundaria has a substantial impact on the decision to migrate. The causal effect is about 0.48 and the estimates are statistically significant at 10% significance level.

The second panel of the table displays the estimates for $E(Y(3) - Y(1)|\boldsymbol{S} = \boldsymbol{s}_{13})$, which is the causal effect of changing education attainment from not completing secundaria to study beyond secundaria on migration. This effect is associated to 2% of the participants. It comprises the subset of compliers that decide to study beyond secundaria due to the incentives offered by Oportunidades. We find the effect to be negative, relatively small, and not statistically significant. The point estimate of the effect ranges from -0.10 to -0.20 when conditioned on baseline variables.

The main feature of this empirical exercise is the use of incentive analysis to assess the question of whether schooling has a non-monotonic effect on the decision to migrate to the US. We focus on the age range most likely to respond to the schooling incentives offered by Oportunidades. Our stylized model enables us to characterize five types that are driven by economic behavior. We are able to evaluate the share of the participants that do respond to Oprtunidades' incentives and evaluate the causal effects of schooling on migration for those who respond to the incentives. We find compelling evidence that completing secundaria increases the likelihood of migration. Our key empirical finding however is the difference between the causal effects. While completing secundaria has a strong effect on the propensity to migrate, studying beyond secundaria does not. These findings corroborate the hypothesis of several works suggesting a negative and non-monotonic selection of migrants regarding education (Behrman et al., 2005; Borjas, 1987; Chiquiar and Hanson, 2005).

Table 5: Causal Effects for Males

Causal Effects	**Model 1**	**Model 2**	**Model 3**	**Model 4**
$E(Y(2) - Y(1) \mid \boldsymbol{S} = \boldsymbol{s}_{21})$	**0.480**	**0.487**	**0.472**	**0.503**
s.e.	0.265	0.261	0.261	0.287
p-val	0.070	0.062	0.071	0.079
$E(Y(3) - Y(1) \mid \boldsymbol{S} = \boldsymbol{s}_{31})$	-0.001	-0.118	-0.135	-0.194
s.e.	0.274	0.320	0.310	0.358
p-val	0.998	0.713	0.662	0.587

This table presents the estimates of the causal effects for males. The estimates are obtained according to the doubly robust orthogonal score estimator described in this section. The estimates comprise four models that differ in terms of the set of baseline variables we seek to control for. Model 1 does not include baseline variables. Model 2 employs age at onset and county migration index. Model 3 adds family characteristics: family members speak indigenous language, number of household members, and number of teenagers. Model 4 includes household assets and house ownership. Estimates consists on the effect, its standard error and the two-sided p-value associated with inference that tests if the effect is equal to zero. Standard errors are computed using the multiplier bootstrap method.

6 Summary and Conclusions

This paper offers a fresh perspective on the identification of causal effects in economic choice models that employ instrumental variables.

We diverge from traditional approaches focusing on identification strategies grounded in novel monotonicity or separability conditions. Instead, we introduce a framework that utilizes revealed preference analysis to translate choice incentives into identification conditions. This method possesses several advantages, notably that identification does not depend on invoking statistical or functional form assumptions. Instead, identification conditions emerge naturally from fundamental economic principles applied to choice incentives. This enhances both the credibility of the underlying assumptions and the clarity of the sources of identification.

The framework is versatile enough to accommodate a wide range of non-trivial identification assumptions, making it applicable in scenarios where traditional IV assumptions may not hold.

We have demonstrated its flexibility by examining well-established examples of choice incentives in the policy evaluation literature, showcasing its adaptability to real-world empirical research. We then examine how popular identification assumption, commonly invoked in the literature of policy evaluation, can be traced back to specific patters of choice incentives. We provide several results that map broad patterns of choice incentives into useful identification

We employ our analytical framework to investigate the migration patterns of impoverished Mexican households to the US. A substantial literature on migration investigates the relationship between education attainment and the likelihood of migration. Seminal work of Borjas (1987, 1994) suggests a negative selection in which those with lowest education benefit the most from moving to the US. On the other hand, Behrman et al. (2005); Chiquiar and Hanson (2005); Hanson (2006) posits a non-monotonic relationship between education and migration, where the fundamental skills such as basic English proficiency taught in secundaria (middle school) increase the propensity to migrate while additional education reduces migration.

We utilize data from Oportunidades, the largest and most significant anti-poverty program in Mexico, to examine whether schooling has a non-monotonic impact on the decision to migrate to the United States. Employing our incentive framework, we identify two causal effects of education on migration for students responding to the schooling incentives provided by Oportunidades.

Specifically, we assess the impact of completing secundaria and the effects of pursuing education beyond this level. Our findings provide compelling evidence that completing secundaria increases the likelihood of migration, whereas advancing schooling beyond middle school has a negative effect on migration. We estimate our model using novel machine learning techniques that assure double robustness of our estimates.

In the broader context of economic research, this paper contributes to the growing body of literature that leverages revealed preference analysis to enhance the identification of causal effects in IV models. Our approach offers a valuable tool for economists grappling with identification issues in diverse and non-standard empirical settings. We make the case that combining economic incentives and classical behavior strengthen the foundations of IV analysis and empowers researchers with a useful tool to evaluate such models.

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Web Appendix

A Model Properties and Identification Analyses

A.1 Proof of Theorem T.1

We use the following auxiliary lemma to prove the result:

Lemma L.1. Let a linear system $\boldsymbol{y} = \boldsymbol{B}\boldsymbol{x}$, where $\boldsymbol{B}$ is a real-valued matrix with full row-rank and ξ be a vector with the same dimension of $\boldsymbol{x}$. Thus, $\xi'\boldsymbol{x}$ is point identified if and only if $\xi'\xi = \xi'\boldsymbol{B}'(\boldsymbol{B}\boldsymbol{B}')^{-1}\boldsymbol{B}\xi$.

Proof. The general solution for $\boldsymbol{x}$ in the system of linear equations represented by $\boldsymbol{y} = \boldsymbol{B}\boldsymbol{x}$ is:[27]

$$\boldsymbol{y} = \boldsymbol{B}\boldsymbol{x} \Rightarrow \boldsymbol{x} = \boldsymbol{B}^{+}\boldsymbol{y} + (\boldsymbol{I} - \boldsymbol{B}^{+}\boldsymbol{B})\boldsymbol{\lambda} \tag{54}$$

where $\boldsymbol{\lambda}$ is an arbitrary real-valued $|\boldsymbol{x}|$-dimension vector, $\boldsymbol{I}$ is an identity matrix of the same dimension and $\boldsymbol{B}^{+}$ is the Moore–Penrose Pseudoinverse of matrix $\boldsymbol{B}$.[28] It follows that a linear combination $\xi'\boldsymbol{x}$ is point identified if and only if $\xi'(\boldsymbol{I} - \boldsymbol{B}^{+}\boldsymbol{B}) = \boldsymbol{0}$. Note that $\boldsymbol{B}^{+}\boldsymbol{B}$ denotes an orthogonal projection since $(\boldsymbol{B}^{+}\boldsymbol{B})' = \boldsymbol{B}^{+}\boldsymbol{B}$ and $(\boldsymbol{B}^{+}\boldsymbol{B})\cdot(\boldsymbol{B}^{+}\boldsymbol{B}) = \boldsymbol{B}^{+}\boldsymbol{B}$ holds. Thus, it is also the case that $\boldsymbol{I}-\boldsymbol{B}^{+}\boldsymbol{B}$ is an orthogonal projection and therefore $(\boldsymbol{I}-\boldsymbol{B}^{+}\boldsymbol{B})(\boldsymbol{I}-(\boldsymbol{B}^{+}\boldsymbol{B}))' = \boldsymbol{I}'-(\boldsymbol{B}^{+}\boldsymbol{B})$. Combining these properties, we have that:

$$\xi'(\boldsymbol{I} - \boldsymbol{B}^{+}\boldsymbol{B}) = \boldsymbol{0} \quad \Leftrightarrow \quad \xi'(\boldsymbol{I} - \boldsymbol{B}^{+}\boldsymbol{B})(\xi'(\boldsymbol{I} - \boldsymbol{B}^{+}\boldsymbol{B}))' = 0 \quad \Leftrightarrow \quad \xi'(\boldsymbol{I} - \boldsymbol{B}^{+}\boldsymbol{B})\xi = 0. \quad \Leftrightarrow \quad \xi'\xi = \xi'\boldsymbol{B}^{+}\boldsymbol{B}\xi.$$

Note that if matrix $\boldsymbol{B}$ has full row rank, the pseudo-inverse matrix is given by $\boldsymbol{B}^{+} = \boldsymbol{B}'(\boldsymbol{B}\boldsymbol{B}')^{-1}$.[29] We can combine these properties to state that $\xi'\boldsymbol{x}$ is point identified if and only if $\xi'\xi = \xi'\boldsymbol{B}'(\boldsymbol{B}\boldsymbol{B}')^{-1}\boldsymbol{B}\xi$. $\square$

Equation (6) establishes the following systems of linear equations:

$$\boldsymbol{Q}_Z(t) \odot \boldsymbol{P}_Z(t) = \boldsymbol{B}_t \left(\boldsymbol{Q}_S(t) \odot \boldsymbol{P}_S\right) \text{ and } \boldsymbol{P}_Z(t) = \boldsymbol{B}_t\boldsymbol{P}_S \text{ for all } t \in \mathcal{T}.$$

We seek to examine the identification of $E(Y(t)|\boldsymbol{S} \in \tilde{\mathcal{S}})$ for some response type set $\tilde{\mathcal{S}} \subset \mathcal{S}$. Let $\boldsymbol{b}(\tilde{\mathcal{S}})$ be the $N_S \times 1$ vector that indicates the types that belongs to set $\tilde{\mathcal{S}}$, namely:

$$\boldsymbol{b}(\tilde{\mathcal{S}}) = \left[\boldsymbol{1}[\boldsymbol{s}_1 \in \tilde{\mathcal{S}}], ..., \boldsymbol{1}[\boldsymbol{s}_{N_S} \in \tilde{\mathcal{S}}]\right]'.$$

Thus we can express $E(Y(t)|\boldsymbol{S} \in \tilde{\mathcal{S}})$ as:

$$E(Y(t)|\boldsymbol{S} \in \tilde{\mathcal{S}}) = \frac{\boldsymbol{b}(\tilde{\mathcal{S}})' \left(\boldsymbol{Q}_S(t) \odot \boldsymbol{P}_S\right)}{\boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{P}_S}. \tag{55}$$

According to Lemma L.1, the criteria for the identification of both the numerator and the denominator of the ratio in (55) is given by $\boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{b}(\tilde{\mathcal{S}}) = \boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{B}_t'(\boldsymbol{B}_t\boldsymbol{B}_t')^{-1}\boldsymbol{B}_t\boldsymbol{b}(\tilde{\mathcal{S}})$. Note that $\boldsymbol{b}(\tilde{\mathcal{S}})$ is an indicator vector. Thus, $\boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{b}(\tilde{\mathcal{S}})$ is simply the cardinality of $\tilde{\mathcal{S}}$, that is, $\boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{b}(\tilde{\mathcal{S}}) = |\tilde{\mathcal{S}}|$. The term $\boldsymbol{B}_t\boldsymbol{b}(\tilde{\mathcal{S}})$ is the sum of the columns of $\boldsymbol{B}_t$ corresponding to the types in $\tilde{\mathcal{S}}$, that is,

[27] See Magnus and Neudecker (1999) for a general discussion of linear systems.

[28] The Moore–Penrose Pseudoinverse $\boldsymbol{B}^{+}$ of matrix $\boldsymbol{B}$ is unique and defined by the following properties: (1) $\boldsymbol{B}\boldsymbol{B}^{+}\boldsymbol{B} = \boldsymbol{B}$; (2) $\boldsymbol{B}^{+}\boldsymbol{B}\boldsymbol{B}^{+} = \boldsymbol{B}^{+}$; (3) $\boldsymbol{B}^{+}\boldsymbol{B} = (\boldsymbol{B}^{+}\boldsymbol{B})'$; and (4) $\boldsymbol{B}\boldsymbol{B}^{+} = (\boldsymbol{B}\boldsymbol{B}^{+})'$.

[29] See Magnus and Neudecker (1999).

$\boldsymbol{B}_t \boldsymbol{b}(\tilde{\mathcal{S}}) = \sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]$. Combining these results, we have the criteria:

$$\boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{b}(\tilde{\mathcal{S}}) = \boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{B}_t'(\boldsymbol{B}_t\boldsymbol{B}_t')^{-1}\boldsymbol{B}_t\boldsymbol{b}(\tilde{\mathcal{S}}) \Leftrightarrow \frac{\left(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\right)' \left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1} \left(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\right)}{|\tilde{\mathcal{S}}|} = 1.$$

This result proves the first part of the theorem. The second part of the theorem employs the general solution of linear systems in (54). If the identification criteria holds, then $P(\boldsymbol{S}\in\tilde{\mathcal{S}})$ can be expressed as:

$$P(\boldsymbol{S}\in\tilde{\mathcal{S}}) = \boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{B}_t^{+}\boldsymbol{P}_Z(t) = \boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{B}_t\left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1}\boldsymbol{P}_Z(t) = \Big(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\Big)'\left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1}\boldsymbol{P}_Z(t).$$

In the same token, the identification of $E(Y(t)|\boldsymbol{S}\in\tilde{\mathcal{S}})P(\boldsymbol{S}\in\tilde{\mathcal{S}})$ is given that:

$$\begin{aligned} E(Y(t)|\boldsymbol{S}\in\tilde{\mathcal{S}})P(\boldsymbol{S}\in\tilde{\mathcal{S}}) = \boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{B}_t^{+}\left(\boldsymbol{Q}_Z(t)\odot\boldsymbol{P}_Z(t)\right) &= \boldsymbol{b}(\tilde{\mathcal{S}})'\boldsymbol{B}_t'\left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1}\left(\boldsymbol{Q}_Z(t)\odot\boldsymbol{P}_Z(t)\right) \\ &= \Big(\sum_{\boldsymbol{s}\in\tilde{\mathcal{S}}} \boldsymbol{B}_t[\cdot,\boldsymbol{s}]\Big)'\left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1}\left(\boldsymbol{Q}_Z(t)\odot\boldsymbol{P}_Z(t)\right). \end{aligned}$$

A.2 Applying Theorem T.1 to LATE

Consider the LATE model where $T\in\{t_0,t_1\}$, $Z\in\{z_0,z_1\}$, and the monotonicity condition $\mathbf{1}[T_i(z_0)=t_1]\le\mathbf{1}[T_i(z_1)=t_1]\forall i$ holds. This model admits three types: never-takers $\boldsymbol{s}_{\text{nt}}=[t_0,t_0]'$, compliers $\boldsymbol{s}_{\text{c}}=[t_0,t_1]'$, and always-takers $\boldsymbol{s}_{\text{at}}=[t_1,t_1]'$. The corresponding response matrix $\boldsymbol{R}$ and the binary matrices $\boldsymbol{B}_{t_0}\equiv\mathbf{1}[\boldsymbol{R}=t_0]$, $\boldsymbol{B}_{t_1}\equiv\mathbf{1}[\boldsymbol{R}=t_1]$ are:

$$\boldsymbol{R} = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_c & \boldsymbol{s}_{at} \end{array} \\ \begin{bmatrix} t_0 & t_0 & t_1 \\ t_0 & t_1 & t_1 \end{bmatrix} \end{array} \begin{array}{c} \\ T(z_0) \\ T(z_1) \end{array} \quad \therefore \quad \boldsymbol{B}_{t_0} = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_c & \boldsymbol{s}_{at} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ T_i(z_0) \\ T_i(z_1) \end{array}, \quad \boldsymbol{B}_{t_1} = \begin{array}{c} \begin{array}{ccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_c & \boldsymbol{s}_{at} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}. \tag{56}$$

It is useful to define the identification criteria $\boldsymbol{H}[t,\boldsymbol{s}]\equiv\boldsymbol{B}_t[\cdot,\boldsymbol{s}]'\left(\boldsymbol{B}_t\boldsymbol{B}_t'\right)^{-1}\boldsymbol{B}_t[\cdot,\boldsymbol{s}]$. According to Theorem T.1, $E(Y(t)|\boldsymbol{S}=\boldsymbol{s})$ is identified if $\boldsymbol{H}[t,\boldsymbol{s}]=1$. The following equation computes the identification criteria $\boldsymbol{H}[t_1,\boldsymbol{s}_c]$ for the treated compliers $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_c)$ of the LATE model:

$$\boldsymbol{H}[t_1,\boldsymbol{s}_c] = \boldsymbol{B}_{t_1}[\cdot,\boldsymbol{s}_c]'\left(\boldsymbol{B}_{t_1}\boldsymbol{B}_{t_1}'\right)^{-1}\boldsymbol{B}_{t_1}[\cdot,\boldsymbol{s}_c] = \begin{bmatrix}0 & 1\end{bmatrix}\left(\begin{bmatrix}0 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\\1 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}0\\1\end{bmatrix} = 1,$$

$\boldsymbol{H}[t_1,\boldsymbol{s}_c]=1$ means that $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_c)$ is identified, and, according to Theorem **T.1**, the identification equation for $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_c)$ is given by:

$$E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_c) = \frac{\left[\boldsymbol{B}_{t_1}[\cdot,\boldsymbol{s}_c]'\left(\boldsymbol{B}_{t_1}\boldsymbol{B}'_{t_1}\right)^{-1}\right]\cdot\left(\boldsymbol{Q}_Z(t_1)\odot\boldsymbol{P}_Z(t_1)\right)}{\left[\boldsymbol{B}_{t_1}[\cdot,\boldsymbol{s}_c]'\left(\boldsymbol{B}_{t_1}\boldsymbol{B}'_{t_1}\right)^{-1}\right]\cdot\left(\boldsymbol{P}_Z(t_1)\right)} \tag{57}$$

$$= \frac{\begin{bmatrix}-1 & 1\end{bmatrix}\cdot\begin{pmatrix}E(Y|T=t_1,Z=z_0)P(T=t_1|Z=z_0)\\ E(Y|T=t_1,Z=z_1)P(T=t_1|Z=z_1)\end{pmatrix}}{\begin{bmatrix}-1 & 1\end{bmatrix}\begin{pmatrix}P(T=t_1|Z=z_0)\\ P(T=t_1|Z=z_1)\end{pmatrix}} \tag{58}$$

$$= \frac{E(Y\cdot D_{t_1}|Z=z_1)-E(Y\cdot D_{t_1}|Z=z_0)}{E(D_{t_1}|Z=z_1)-E(D_{t_1}|Z=z_0),}, \tag{59}$$

where $D_{t_1} \equiv \mathbf{1}[T=t_1]$. The parameter can be estimated by a 2SLS regression that uses Z to instrument the effect of the endogenous choice indicator D_{t_1} on the outcome variable $Y\cdot D_{t_1}$. The the following *Identification Matrix* displays the value of the identification criteria $\boldsymbol{H}[t,\boldsymbol{s}]$ for all $(t,\boldsymbol{s})\in\{t_0,t_1\}\times\{\boldsymbol{s}_n,\boldsymbol{s}_c,\boldsymbol{s}_1\}$ of the LATE model:

$$\textbf{LATE Identification Matrix} \quad \boldsymbol{H} \quad = \quad \begin{array}{cc} \begin{array}{ccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_c & \boldsymbol{s}_{at} \end{array} & \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} t_0 \\ t_1 \end{array} \end{array} \tag{60}$$

The matrix indicates that the identification status of six counterfactual outcomes. Four counterfactual outcomes are identified: $E(Y(t_0)|\boldsymbol{S}=\boldsymbol{s}_c)$ and $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_c)$ for compliers, $E(Y(t_0)|\boldsymbol{S}=\boldsymbol{s}_{nt})$ for never-takes, and $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_a)$ for always-taker. Neither $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_{nt})$ or $E(Y(t_0)|\boldsymbol{S}=\boldsymbol{s}_{at})$ are identified, indeed, they are not even defined. The expression that identifies the counterfactual outcome for treated compliers, $E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_c)$ is presented in equations (57)–(59). The remaining expressions according to Theorem **T.1** are displayed below:

$$E(Y(t_0)|\boldsymbol{S}=\boldsymbol{s}_{nt}) = \frac{\boldsymbol{B}_{t_0}[\cdot,\boldsymbol{s}_{nt}]'\left(\boldsymbol{B}_{t_0}\boldsymbol{B}'_{t_0}\right)^{-1}\cdot(\boldsymbol{Q}_Z(t_0)\odot\boldsymbol{P}_Z(t_0))}{\boldsymbol{B}_{t_0}[\cdot,\boldsymbol{s}_{nt}]'\left(\boldsymbol{B}_{t_0}\boldsymbol{B}'_{t_0}\right)^{-1}\cdot\boldsymbol{P}_Z(t_0)} = \frac{E(Y\cdot D_{t_0}|Z=z_1)}{E(D_{t_0}|Z=z_1),},$$

$$E(Y(t_1)|\boldsymbol{S}=\boldsymbol{s}_{at}) = \frac{\boldsymbol{B}_{t_1}[\cdot,\boldsymbol{s}_{at}]'\left(\boldsymbol{B}_{t_1}\boldsymbol{B}'_{t_1}\right)^{-1}\cdot(\boldsymbol{Q}_Z(t_1)\odot\boldsymbol{P}_Z(t_1))}{\boldsymbol{B}_{t_1}[\cdot,\boldsymbol{s}_{at}]'\left(\boldsymbol{B}_{t_1}\boldsymbol{B}'_{t_1}\right)^{-1}\cdot\boldsymbol{P}_Z(t_1)} = \frac{E(Y\cdot D_{t_1}|Z=z_0)}{E(D_{t_1}|Z=z_0),},$$

$$E(Y(t_0)|\boldsymbol{S}=\boldsymbol{s}_c) = \frac{\boldsymbol{B}_{t_0}[\cdot,\boldsymbol{s}_c]'\left(\boldsymbol{B}_{t_0}\boldsymbol{B}'_{t_0}\right)^{-1}\cdot(\boldsymbol{Q}_Z(t_0)\odot\boldsymbol{P}_Z(t_0))}{\boldsymbol{B}_{t_0}[\cdot,\boldsymbol{s}_c]'\left(\boldsymbol{B}_{t_0}\boldsymbol{B}'_{t_0}\right)^{-1}\cdot\boldsymbol{P}_Z(t_0)} = \frac{E(Y\cdot D_{t_0}|Z=z_0)-E(Y\cdot D_{t_0}|Z=z_1)}{E(D_{t_0}|Z=z_0)-E(D_{t_0}|Z=z_1),}.$$

We can combine the identification equations for the treated and untreated compliers to obtain the well-known LATE expression:

$$E(Y(t_1)-Y(t_0)) = \frac{E(Y|Z=z_1)-E(Y|Z=z_0)}{P(T=t_1|Z=z_1)-P(T=t_0|Z=z_1)}.$$

Now consider the LATE model in which we relax the monotonicity condition. In this case, the response matrix and the corresponding identification matrix are:

$$\boldsymbol{R} \quad = \quad \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_c & \boldsymbol{s}_{at} & \boldsymbol{s}_d \\ \end{array} \hspace{-14em} \begin{array}{l} \\ \begin{bmatrix} t_0 & t_0 & t_1 & t_1 \\ t_0 & t_1 & t_1 & t_0 \end{bmatrix} \begin{array}{l} T_i(z_0) \\ T_i(z_1) \end{array} \end{array} \Rightarrow \boldsymbol{H} \quad = \quad \begin{array}{l} \begin{array}{cccc} \boldsymbol{s}_{nt} & \boldsymbol{s}_c & \boldsymbol{s}_{at} & \boldsymbol{s}_d \end{array} \\ \begin{bmatrix} 2/3 & 2/3 & 0 & 2/3 \\ 0 & 2/3 & 2/3 & 2/3 \end{bmatrix} \begin{array}{l} t_0 \\ t_1 \end{array} \end{array}$$

Note that none of the elements of the identification matrix are equal to one, which indicates that there are no point-identified counterfactuals when the monotonicity condition is relaxed.

A.3 Using Revealed Preference Analysis to Generate Choice Restrictions

Our choice model stems from a classical economic framework where the potential choice of agent i for a fixed IV-value z is characterized by the following utility maximization problem:

$$\text{Choice Equation}: \quad T_i(z) = \text{argmax}_{t \in \mathcal{T}} \left(\max_{\boldsymbol{g} \in \mathcal{B}_i(Z_i, t)} u_i(t, \boldsymbol{g}) \right). \tag{61}$$

The real-valued utility function $u_i : \mathcal{T} \times \mathbb{R}_+^K$ represents the rational preferences of agent i towards the bundle $(t, \boldsymbol{g})$ where t is the treatment status and $\boldsymbol{g}$ is a K-dimensional vector of unobserved consumption goods. The set $\mathcal{B}_i(z, t) \subset \mathbb{R}_+^K$ stands for the potential budget set of consumption goods $\boldsymbol{g}$ of agent i when the treatment is *fixed* to $t \in \mathcal{T}$ and the instrument is *fixed* to the value $z \in \mathcal{Z}$. The budget set is broadly interpreted to encompass various decisions extending beyond traditional consumption goods. It can include decisions regarding education attainment, neighborhood selection, and time allocation depending on the empirical setting under examination.

The incentive matrix $\boldsymbol{L}$ characterises budget set relationships in which bigger incentives correspond to larger budget sets for a given a choice t :

$$\text{Budget Relationships:} \quad \boldsymbol{L}[z, t] \leq \boldsymbol{L}[z', t] \Rightarrow \mathcal{B}_i(z, t) \subseteq \mathcal{B}_i(z', t). \tag{62}$$

To put in context, consider the LATE model where $T = t_1$ denotes college enrollment and $T = t_0$ denotes no college. $Z \in$ is a randomly assigned tuition discount, z_1 if the discount is granted and z_0 if not. The LATE incentive matrix yields the following budget set relations:

$$\underbrace{\boldsymbol{L} = \begin{array}{l} \begin{array}{cc} t_0 & t_1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{l} z_0 \\ z_1 \end{array} \end{array}}_{\text{LATE Incentive Matrix}} \quad \Rightarrow \quad \underbrace{\begin{array}{l} \mathcal{B}_i(z_0, t_0) = \mathcal{B}_i(z_1, t_0) \\ \mathcal{B}_i(z_0, t_1) \subset \mathcal{B}_i(z_1, t_1) \end{array}}_{\text{Implied Budget Set Relationships}} \tag{63}$$

The budget set equality $\mathcal{B}_i(z_0, t_0) = \mathcal{B}_i(z_1, t_0)$ implies that when the choice is set to no college t_0, the tuition discount is irrelevant. Conversely, $\mathcal{B}_i(z_0, t_1) \subset \mathcal{B}_i(z_1, t_1)$ suggests that the tuition discount increases agent i's budget if they choose to attend college.

Budget relationships enable us to use the Weak Axiom of Revealed Preference (WARP) of Richter (1971). Bundles $(t, \boldsymbol{g})$ and $(t', \boldsymbol{g}')$ are said to be available given z if $\boldsymbol{g} \in \mathcal{B}_i(z, t)$ and $\boldsymbol{g}' \in \mathcal{B}_i(z, t')$. If bundle $(t, \boldsymbol{g})$ is chosen by agent i when $(t, \boldsymbol{g})$ and $(t', \boldsymbol{g}')$ are available, then $(t, \boldsymbol{g})$ is said to be directly and strictly revealed preferred to $(t', \boldsymbol{g}')$, that is, $(t, \boldsymbol{g}) \succ^d_{i,z} (t', \boldsymbol{g}')$. WARP states that if $(t, \boldsymbol{g})$ revealed preferred to $(t', \boldsymbol{g}')$ under $z \in \mathcal{Z}$, then $(t', \boldsymbol{g}')$ cannot be revealed preferred to $(t, \boldsymbol{g})$ under $z' \in \mathcal{Z} \setminus \{z\}$. Notationally, we write that:

$$\text{WARP:} \quad (t, \boldsymbol{g}) \succ^d_{i,z} (t', \boldsymbol{g}') \Rightarrow (t', \boldsymbol{g}') \not\succ^d_{i,z'} (t, \boldsymbol{g}). \tag{64}$$

The following lemma uses WARP and the budget relations (62) to translate incentives into choice restrictions.

Lemma L.2. Let a choice model with an incentive matrix $\boldsymbol{L}$. Under the budget relationships (62) and WARP (64), the following choice rule holds:

$$\text{WARP Rule:} \quad \text{If } T_i(z) = t, \text{ and } \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \leq 0 \leq \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \text{ then } T_i(z') \neq t'. \tag{65}$$

Proof. $T_i(z) = t$ implies that there exists $\boldsymbol{g} \in \mathcal{B}_i(z, t)$ such that $(t, \boldsymbol{g}) \succ^d_{i,z} (t', \boldsymbol{g}')$ for all $\boldsymbol{g}' \in \mathcal{B}_i(z, t')$. The inequality $0 \leq \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t]$ implies that the budget set associated with t increases as we move from z to z', $\mathcal{B}_i(z, t) \subseteq \mathcal{B}_i(z', t)$. Thus the bundle $(t, \boldsymbol{g})$ remains available under z'. On the other hand, $\boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \leq 0$ implies that the budget set associated with t' decreases as we move from z to z', $\mathcal{B}_i(z', t') \subseteq \mathcal{B}_i(z, t')$. Thus any bundle $(t', \boldsymbol{g}'')$ that is available under z', $(t', \boldsymbol{g}''); g'' \in \mathcal{B}_i(z', t')$ were also available under z. Thus, according to WARP, agent i still prefers $(t, \boldsymbol{g})$ to any $(t', \boldsymbol{g}''); g'' \in \mathcal{B}_i(z', t')$, that is, $(t', \boldsymbol{g}'') \not\succ^d_{i,z'} (t, \boldsymbol{g})$. This implies that agent i does not choose t' under z', $T_i(z') \neq t'$. $\square$

As mentioned in the main paper, applying the WARP Rule 65 to LATE incentives (63) yields the choice restriction $T_i(z_0) = t_1 \Rightarrow T_i(z_1) \neq t_0$, which means that if the student chooses college under no incentives, it will not choose otherwise when incentives to enroll in college are offered.

It is possible to exploit additional economic choice behaviors that enable us to enhance the WARP rule. For instance, consider the choice of a college student who debates between two majors: electrical or mechanical engineering. Suppose the student chooses electrical over mechanical engineering under no tuition discount. In that case, it is natural to assume that the student will maintain choice when granted a tuition discount that applies to both majors. This behavior is captured by the condition called *Normal Choice*:

$$\text{Normal Choice: } t \succ_{i,z} t' \text{ and } \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] = \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \text{ then } t \succ_{i,z'} t' \text{ holds,} \tag{66}$$

where $t \succ_{i,z} t'$ means that there is $\boldsymbol{g} \in \mathcal{B}_i(z, t)$ such that $(t, \boldsymbol{g}) \succ_{i,z} (t', \boldsymbol{g}')$ for all $\boldsymbol{g}' \in \mathcal{B}_i(z, t')$. Normal Choice states that if an agent i prefers t instead of t' under z, and if the change in incentives for choosing either t or t' is the same under z', then agent i maintains its preference of t over t' under z'.[30] WARP and Normal Choice (66) yield the following choice rule:

[30] Normal Choice is a no-crossing condition on the ranking of choice preferences that maintains the relative rank of two choices that share the same incentives. The normal choice is related to the notion of normal goods. Consider an agent that debates between two goods a and b. Suppose a discount of d dollars is applied to both goods. This discount can be understood as an increase in income of d dollars since the agent will benefit from it regardless of his choice. An increase in income does not decrease the consumption of a normal good. If the agent decides to buy a under no discount, it will continue to consume one unit of good a when the discount is available.

Lemma L.3. Let a choice model with an incentive matrix $\boldsymbol{L}$. Under the budget relationships (62), WARP (64), and Normal Choice (66), the following choice rule holds:

$$\text{Choice Rule:} \quad \text{If } T_i(z) = t \text{ and } \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \leq \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \text{ then } T_i(z') \neq t'. \tag{67}$$

Proof. Let $\delta_t \equiv \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t]$ and $\delta_{t'} \equiv \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t']$, where $\delta_{t'} \leq \delta_t$. Note that we can set $\boldsymbol{L}[z, t] = \boldsymbol{L}[z, t'] = 0$, $\boldsymbol{L}[z', t] = \delta_t$, and $\boldsymbol{L}[z', t'] = \delta_{t'}$ without loss of generality. Note also that $T_i(z) = t$ means that $t \succ_{i,z} t'$. Now consider an auxiliary instrument z^* that sets $\boldsymbol{L}[z^*, t] = \boldsymbol{L}[z^*, t'] = \delta_{t'}$. We first examine the change from z to z^*. In this case, we have that $\boldsymbol{L}[z^*, t'] - \boldsymbol{L}[z, t'] = \boldsymbol{L}[z^*, t] - \boldsymbol{L}[z, t] = \delta_{t'}$. According to Normal Choice (66), we have that $t \succ_{i,z^*} t'$. Now consider the change from z^* to z'. The inequality $\delta_{t'} \leq \delta_t$ implies that: $\boldsymbol{L}[z', t'] - \boldsymbol{L}[z^*, t'] = 0 \leq \boldsymbol{L}[z', t] - \boldsymbol{L}[z^*, t]$. By WARP Rule (65), we have that $t \succ_{i,z'} t'$ and therefore $T_i(z') \neq t'$. □

As mentioned, the Choice Rule highlights a cornerstone principle of rational choice theory, which posits that an individual's preferences will remain consistent unless there is a compelling incentive to choose otherwise. Specifically, if an agent chooses t over t' when presented with z-incentives, and if z'-incentives are at least as persuasive for choice t as they are for t', then the agent will not choose t' over t.

A.4 Additional Analyses of the IV Model in Example E.3

The incentive matrix of Example **E.3** is:

$$\boldsymbol{L} = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} t_0 & t_1 & t_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_0 \\ z_1 \\ z_2 \end{array} \tag{68}$$

The incentive matrix (68) justifies two monotonicity conditions:

$$\mathbf{1}[T_i(z_0) = t_1] \leq \mathbf{1}[T_i(z_1) = t_1] \tag{69}$$

$$\mathbf{1}[T_i(z_0) = t_2] \leq \mathbf{1}[T_i(z_2) = t_2]. \tag{70}$$

These monotonicity conditions eliminate 12 out of the 27 possible response types as described in Panel B of Table A.1.

The remaining 15 response types are displayed in response matrix below:

$$\boldsymbol{R} = \begin{array}{c} \begin{array}{ccccccccccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 & \boldsymbol{s}_{10} & \boldsymbol{s}_{11} & \boldsymbol{s}_{12} & \boldsymbol{s}_{13} & \boldsymbol{s}_{14} & \boldsymbol{s}_{15} \end{array} \\ \begin{bmatrix} t_0 & t_0 & t_0 & t_0 & t_0 & t_0 & t_0 & t_0 & t_0 & t_2 & t_1 & t_1 & t_2 & t_2 & t_2 \\ t_0 & t_0 & t_0 & t_1 & t_1 & t_1 & t_2 & t_2 & t_2 & t_2 & t_1 & t_1 & t_0 & t_1 & t_2 \\ t_0 & t_1 & t_2 & t_0 & t_1 & t_2 & t_0 & t_1 & t_2 & t_0 & t_1 & t_2 & t_2 & t_2 & t_2 \end{bmatrix} \end{array} \begin{array}{c} \\ T_i(z_0) \\ T_i(z_1) \\ T_i(z_2) \end{array} \tag{71}$$

The response matrix is then used as input to generate the identification matrix $\boldsymbol{H}$, which is a $N_{\times} N_T$-dimensional matrix whose elements are given by $\boldsymbol{H}[t, \boldsymbol{s}] \equiv \boldsymbol{B}_t[\cdot, \boldsymbol{s}]' \left(\boldsymbol{B}_t \boldsymbol{B}_t'\right)^{-1} \boldsymbol{B}_t[\cdot, \boldsymbol{s}]$.

Table A.1: Elimination of Response types of the IV Model in Example **E.3**

Panel A	**All 27 Possible Response types**																										
Counterfactual Choices	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$T_1(z_0)$	t_0	t_0	t_0	t_0	t_0	t_0	t_0	t_0	t_0	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2
$T_1(z_1)$	t_0	t_0	t_0	t_1	t_1	t_1	t_2	t_2	t_2	t_0	t_0	t_0	t_1	t_1	t_1	t_2	t_2	t_2	t_0	t_0	t_0	t_1	t_1	t_1	t_2	t_2	t_2
$T_1(z_2)$	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2	t_0	t_1	t_2
Panel B	**Response type Eliminated by Monotonicity Conditions (69)–(70)**																										
Condition 1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
Condition 2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✓	✗	✗	✓	✗	✗	✓
Not Eliminated	1	2	3	4	5	6	7	8	9				13	14	15						21			24			27
Panel C	**Response type Eliminated by Revealed Preference Analysis**																										
Restriction 1	✓	✗	✓	✓	✗	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Restriction 2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
Restriction 3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✓	✗	✗	✓
Restriction 4	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓	✗	✗	✗	✓	✓	✓	✓	✓	✓	✗	✗	✓
Restriction 5	✓	✗	✓	✓	✗	✓	✓	✗	✓	✓	✗	✓	✓	✓	✓	✓	✗	✓	✓	✗	✓	✓	✗	✓	✓	✗	✓
Not Eliminated	1		3	4		6								14	15									24			27

Panel A lists the 27 possible response types that the response variable $S_1 = [T_1(z_0)\,,\, T_1(z_1)\,,\, T_1(z_2)]$ can take. Rows present the counterfactual choices an agent i could choose if it were assigned to z_0, z_1, and z_2 respectively. Columns present all the values of response type as choices range over $\text{supp}(T) = \{t_0, t_1, t_2\}$. Panel B describes an elimination process based on the two monotonicity conditions (69)–(70). These criteria are also stated in Panel B′ below. Panel C describes an elimination process based on the seven choice restrictions generated by the revealed preference analysis. These choice restrictions are also displayed in Panel C′ below.

Panel B′ – Monotonicity Relations

Monotonicity Condition 1	$\mathbf{1}[T_i(z_0) = t_1]$	$\leq$	$\mathbf{1}[T_i(z_1) = t_1]$
Monotonicity Condition 2	$\mathbf{1}[T_i(z_0) = t_2]$	$\leq$	$\mathbf{1}[T_i(z_2) = t_2]$

Panel C′ – Choice Restrictions

Choice Restriction 1	$T_i(z_0) = t_0$	$\Rightarrow$	$T_i(z_1) \neq t_2$	and	$T_i(z_2) \neq t_1$
Choice Restriction 2	$T_i(z_0) = t_1$	$\Rightarrow$	$T_i(z_1) = t_1$	and	$T_i(z_2) \neq t_0$
Choice Restriction 3	$T_i(z_0) = t_2$	$\Rightarrow$	$T_i(z_1) \neq t_0$	and	$T_i(z_2) = t_2$
Choice Restriction 4	$T_i(z_1) = t_2$	$\Rightarrow$	$T_i(z_0) = t_2$	and	$T_i(z_2) = t_2$
Choice Restriction 5	$T_i(z_2) = t_1$	$\Rightarrow$	$T_i(z_0) = t_1$	and	$T_i(z_1) = t_1$

Check mark ✓ indicates that the response type displayed by the top column of the table does not violate the choice restriction denoted by the panel row. Cross sign ✗indicates that the response type violates the choice restriction and should be eliminated. The last row in each panel presents the response types that survive the elimination process.

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 & \boldsymbol{s}_{10} & \boldsymbol{s}_{11} & \boldsymbol{s}_{12} & \boldsymbol{s}_{13} & \boldsymbol{s}_{14} & \boldsymbol{s}_{15} \end{array} \\
\boldsymbol{H} = & \begin{bmatrix} 11/27 & 8/27 & 8/27 & 8/27 & 5/27 & 5/27 & 8/27 & 5/27 & 5/27 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 4/13 & 0 & 7/26 & 11/26 & 7/26 & 0 & 4/13 & 0 & 0 & 15/26 & 15/26 & 0 & 7/26 & 0 \\ 0 & 0 & 16/75 & 0 & 0 & 16/75 & 19/75 & 19/75 & 31/75 & 34/75 & 0 & 16/75 & 1/3 & 1/3 & 8/25 \end{bmatrix}
\end{array}
$$

Note that none of the entries of the identification matric is equal to one. According to **T.1**, this means that not a single counterfactual outcome of the type $E(Y(t)|\boldsymbol{S}\boldsymbol{s}); (t,\boldsymbol{s}) \in \{t_0, t_1, t_2\} \times \{\boldsymbol{s}_1, ..., \boldsymbol{s}_{15}\}$ is identified. We conclude that the elimination of response type due to the monotonicity conditions (69)–(70) is not sufficient to point-identify counterfactual outcome.

Revealed preference analysis is more effective in eliminating types than the monotonicity conditions. Table A.2 applies choice rule (9) to the incentive matrix (68). There are 22 binding restrictions. Table A.3 summarise these 22 choice restrictions of Table A.2 into the five restrictions,[31] and Panel C of Table (A.1) shows that these five restrictions eliminate 19 out of the 27 possible response types.

Table A.2: Choice Restrictions of Example **E.3** Due to Revealed Preference Analysis

#	Revealed Choice	Incentive Inequalities	Choice Statement
	$T_i(z) = t$	$\boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \leq 0 \leq \boldsymbol{L}[z',t] - \boldsymbol{L}[z,t]$	$T(z') \neq t'$
1	$T_i(z_0) = t_0,$	$\boldsymbol{L}[z_2,t_1] - \boldsymbol{L}[z_0,t_1] = 0 \leq 0 = \boldsymbol{L}[z_2,t_0] - \boldsymbol{L}[z_0,t_0]$	$T_i(z_2) \neq t_1$
2	$T_i(z_0) = t_0,$	$\boldsymbol{L}[z_1,t_2] - \boldsymbol{L}[z_0,t_2] = 0 \leq 0 = \boldsymbol{L}[z_1,t_0] - \boldsymbol{L}[z_0,t_0]$	$T_i(z_1) \neq t_2$
3	$T_i(z_0) = t_1,$	$\boldsymbol{L}[z_1,t_0] - \boldsymbol{L}[z_0,t_0] = 0 \leq 1 = \boldsymbol{L}[z_1,t_1] - \boldsymbol{L}[z_0,t_1]$	$T_i(z_1) \neq t_0$
4	$T_i(z_0) = t_1,$	$\boldsymbol{L}[z_2,t_0] - \boldsymbol{L}[z_0,t_0] = 0 \leq 0 = \boldsymbol{L}[z_2,t_1] - \boldsymbol{L}[z_0,t_1]$	$T_i(z_2) \neq t_0$
5	$T_i(z_0) = t_1,$	$\boldsymbol{L}[z_1,t_2] - \boldsymbol{L}[z_0,t_2] = 0 \leq 1 = \boldsymbol{L}[z_1,t_1] - \boldsymbol{L}[z_0,t_1]$	$T_i(z_1) \neq t_2$
6	$T_i(z_0) = t_2,$	$\boldsymbol{L}[z_1,t_0] - \boldsymbol{L}[z_0,t_0] = 0 \leq 0 = \boldsymbol{L}[z_1,t_2] - \boldsymbol{L}[z_0,t_2]$	$T_i(z_1) \neq t_0$
7	$T_i(z_0) = t_2,$	$\boldsymbol{L}[z_2,t_0] - \boldsymbol{L}[z_0,t_0] = 0 \leq 1 = \boldsymbol{L}[z_2,t_2] - \boldsymbol{L}[z_0,t_2]$	$T_i(z_2) \neq t_0$
8	$T_i(z_0) = t_2,$	$\boldsymbol{L}[z_2,t_1] - \boldsymbol{L}[z_0,t_1] = 0 \leq 1 = \boldsymbol{L}[z_2,t_2] - \boldsymbol{L}[z_0,t_2]$	$T_i(z_2) \neq t_1$
9	$T_i(z_1) = t_0,$	$\boldsymbol{L}[z_0,t_1] - \boldsymbol{L}[z_1,t_1] = -1 \leq 0 = \boldsymbol{L}[z_0,t_0] - \boldsymbol{L}[z_1,t_0]$	$T_i(z_0) \neq t_1$
10	$T_i(z_1) = t_0,$	$\boldsymbol{L}[z_2,t_1] - \boldsymbol{L}[z_1,t_1] = -1 \leq 0 = \boldsymbol{L}[z_2,t_0] - \boldsymbol{L}[z_1,t_0]$	$T_i(z_2) \neq t_1$
11	$T_i(z_1) = t_0,$	$\boldsymbol{L}[z_0,t_2] - \boldsymbol{L}[z_1,t_2] = 0 \leq 0 = \boldsymbol{L}[z_0,t_0] - \boldsymbol{L}[z_1,t_0]$	$T_i(z_0) \neq t_2$
12	$T_i(z_1) = t_2,$	$\boldsymbol{L}[z_0,t_0] - \boldsymbol{L}[z_1,t_0] = 0 \leq 0 = \boldsymbol{L}[z_0,t_2] - \boldsymbol{L}[z_1,t_2]$	$T_i(z_0) \neq t_0$
13	$T_i(z_1) = t_2,$	$\boldsymbol{L}[z_2,t_0] - \boldsymbol{L}[z_1,t_0] = 0 \leq 1 = \boldsymbol{L}[z_2,t_2] - \boldsymbol{L}[z_1,t_2]$	$T_i(z_2) \neq t_0$
14	$T_i(z_1) = t_2,$	$\boldsymbol{L}[z_0,t_1] - \boldsymbol{L}[z_1,t_1] = -1 \leq 0 = \boldsymbol{L}[z_0,t_2] - \boldsymbol{L}[z_1,t_2]$	$T_i(z_0) \neq t_1$
15	$T_i(z_1) = t_2,$	$\boldsymbol{L}[z_2,t_1] - \boldsymbol{L}[z_1,t_1] = -1 \leq 1 = \boldsymbol{L}[z_2,t_2] - \boldsymbol{L}[z_1,t_2]$	$T_i(z_2) \neq t_1$
16	$T_i(z_2) = t_0,$	$\boldsymbol{L}[z_0,t_1] - \boldsymbol{L}[z_2,t_1] = 0 \leq 0 = \boldsymbol{L}[z_0,t_0] - \boldsymbol{L}[z_2,t_0]$	$T_i(z_0) \neq t_1$
17	$T_i(z_2) = t_0,$	$\boldsymbol{L}[z_0,t_2] - \boldsymbol{L}[z_2,t_2] = -1 \leq 0 = \boldsymbol{L}[z_0,t_0] - \boldsymbol{L}[z_2,t_0]$	$T_i(z_0) \neq t_2$
18	$T_i(z_2) = t_0,$	$\boldsymbol{L}[z_1,t_2] - \boldsymbol{L}[z_2,t_2] = -1 \leq 0 \leq 0 = \boldsymbol{L}[z_1,t_0] - \boldsymbol{L}[z_2,t_0]$	$T_i(z_1) \neq t_2$
19	$T_i(z_2) = t_1,$	$\boldsymbol{L}[z_0,t_0] - \boldsymbol{L}[z_2,t_0] = 0 \leq 0 = \boldsymbol{L}[z_0,t_1] - \boldsymbol{L}[z_2,t_1]$	$T_i(z_0) \neq t_0$
20	$T_i(z_2) = t_1,$	$\boldsymbol{L}[z_1,t_0] - \boldsymbol{L}[z_2,t_0] = 0 \leq 1 = \boldsymbol{L}[z_1,t_1] - \boldsymbol{L}[z_2,t_1]$	$T_i(z_1) \neq t_0$
21	$T_i(z_2) = t_1,$	$\boldsymbol{L}[z_0,t_2] - \boldsymbol{L}[z_2,t_2] = -1 \leq 0 = \boldsymbol{L}[z_0,t_1] - \boldsymbol{L}[z_2,t_1]$	$T_i(z_0) \neq t_2$
22	$T_i(z_2) = t_1,$	$\boldsymbol{L}[z_1,t_2] - \boldsymbol{L}[z_2,t_2] = -1 \leq 1 = \boldsymbol{L}[z_1,t_1] - \boldsymbol{L}[z_2,t_1]$	$T_i(z_1) \neq t_2$

This table displays the binding choice restrictions generated by choice rule (9) to the incentive matrix of Example **E.3**.

[31] The remaining restrictions do not eliminate any additional response types that is not already covered by these five restrictions.

Table A.3: Summary of Choice Restrictions generated by applying Choice Rule (9) to Example **E.3**

#	Choice Restrictions				
1,2	$T_i(z_0) = t_0$	$\Rightarrow$	$T_i(z_1) \neq t_2$	and	$T_i(z_2) \neq t_1$
3,4,5	$T_i(z_0) = t_1$	$\Rightarrow$	$T_i(z_1) = t_1$	and	$T_i(z_2) \neq t_0$
6,7,8	$T_i(z_0) = t_2$	$\Rightarrow$	$T_i(z_1) \neq t_0$	and	$T_i(z_2) = t_2$
12,13,14,15	$T_i(z_1) = t_2$	$\Rightarrow$	$T_i(z_0) = t_2$	and	$T_i(z_2) = t_2$
19,20,21,22	$T_i(z_2) = t_1$	$\Rightarrow$	$T_i(z_0) = t_1$	and	$T_i(z_1) = t_1$

The eight response types that survive the elimination process are displayed in the response matrix below:

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 \end{array} \\
\boldsymbol{R} = & \begin{bmatrix} t_0 & t_0 & t_0 & t_0 & t_1 & t_1 & t_2 & t_2 \\ t_0 & t_0 & t_1 & t_1 & t_1 & t_1 & t_1 & t_2 \\ t_0 & t_2 & t_0 & t_2 & t_1 & t_2 & t_2 & t_2 \end{bmatrix} \begin{array}{l} T_i(z_0) \\ T_i(z_1) \\ T_i(z_2) \end{array}
\end{array}$$

The corresponding identification matrix is given by:

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 \end{array} \\
\boldsymbol{H} = & \begin{bmatrix} 3/4 & 3/4 & 3/4 & 3/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1 & 1 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 1 & 1 \end{bmatrix} \begin{array}{l} t_0 \\ t_1 \\ t_2 \end{array}
\end{array}$$

The entries of the identification matrix show that four counterfactual outcomes are point-identified, namely, $E(Y(t_1)|\boldsymbol{S} = \boldsymbol{s}_5)$, $E(Y(t_1)|\boldsymbol{S} = \boldsymbol{s}_6)$, $E(Y(t_2)|\boldsymbol{S} = \boldsymbol{s}_7)$, and $E(Y(t_2)|\boldsymbol{S} = \boldsymbol{s}_8)$.

A.5 Analysing Choice Restrictions of the IV Model in Example E.5

Let $T \in \{0, 2, 4\}$ represent the number of years of the college degree and let the instrument be $Z = (Z_2, Z_4) \in \{0, 1\}^2$, where Z_2 and Z_4 indicate the proximity to two-year and four-year colleges, respectively. We use $T(z_2, z_4)$ for the counterfactual choice. The response vector is $\boldsymbol{S} = [T(0,0), T(0,1), T(1,0), T(1,1)]'$, which can take 81 potential response types. The incentive matrix of this choice model is:

$$\begin{array}{cc}
 & \begin{array}{ccc} 0 & 2 & 4 \end{array} \quad (z_2, z_4) \\
\boldsymbol{L} = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array}
\end{array}$$

We seek to investigate the choice restrictions generated by choice rule (9). The rule states that:

$$\textbf{Choice Rule:} \quad \text{If } T_i(z) = t \text{ and } \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \leq \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \text{ then } T_i(z') \neq t'.$$

The rule applies to all combinations of (t, t') of treatments in $\{0, 2, 4\}$ and all the combination of IV-values (z, z') in $\{0, 1\}^2$. There are three treatments, which yield six combinations of treatments. There are four IV-values, which yield 12 combinations of IV-value. Thus, we must apply the rule $6 \times 12 = 72$ times.

Some of these applications generate incentive relationships that adhere to the rule criteria. For instance, consider the comparison between the IV-values $z = (0, 0)$ (first row of $\boldsymbol{L}$) and $z' = (0, 1)$ (second role of $\boldsymbol{L}$). If we set $t = 0$ against $t' = 2$ we have the following incentive condition:

$$\boldsymbol{L}[(0,1), 2] - \boldsymbol{L}[(0,1), 2] = 1 \not\leq 0 = \boldsymbol{L}[(0,1), 0] - \boldsymbol{L}[(0,0), 0]$$

Thus, $T_i(0, 0) = 0$ does not imply that $T_i(0, 1) \neq 2$. On the other hand, if we set $t = 4$ against $t' = 2$ we have that:

$$\boldsymbol{L}[(0,1), 2] - \boldsymbol{L}[(0,1), 2] = 1 \leq 1 = \boldsymbol{L}[(0,1), 4] - \boldsymbol{L}[(0,0), 4]$$

Thus the choice restriction $T_i(0, 0) = 4 \Rightarrow T_i(0, 1) \neq 2$ holds. The following table indicates all the binding incentive conditions across each of the possible combinations among $t, t' \in \{0, 2, 4\}$ and $z, z' in \{0, 1\}^2$:

Table A.4: Binding Incentives in Example E.5

		t	0	0	2	2	4	4
		t'	2	4	0	4	0	2
	z	z'	1	2	3	4	5	6
1	$(0,0)$	$(0,1)$	✓	✗	✓	✗	✓	✓
2	$(0,0)$	$(1,0)$	✗	✓	✓	✓	✓	✗
3	$(0,0)$	$(1,1)$	✗	✗	✓	✓	✓	✓
4	$(0,1)$	$(0,0)$	✓	✓	✓	✓	✗	✗
5	$(0,1)$	$(1,0)$	✗	✓	✓	✓	✗	✗
6	$(0,1)$	$(1,1)$	✗	✓	✓	✓	✓	✗
7	$(1,0)$	$(0,0)$	✓	✓	✗	✗	✓	✓
8	$(1,0)$	$(0,1)$	✓	✗	✗	✗	✓	✓
9	$(1,0)$	$(1,1)$	✓	✗	✓	✗	✓	✓
10	$(1,1)$	$(0,0)$	✓	✓	✗	✓	✗	✓
11	$(1,1)$	$(0,1)$	✓	✓	✗	✗	✓	✓
12	$(1,1)$	$(1,0)$	✓	✓	✓	✓	✗	✗

This table indicates weather the incentive condition $\boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \leq \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t]$ for all combinations of $t, t' \in \{0, 2, 4\}$ and all $z, z' \in \{0, 1\}^2$. Incentive conditions that hold are denoted by ✓. Those which do not hold are denoted by ✗.

The content of the table is divided into twelve blocks separated by horizontal and vertical lines. The case $T_i(0,0) = 0$ is examined in the first block (rows 1–3, columns 1–2). This block contains two binding incentive conditions, which yield two choice restrictions. The first one is $T_i(0,0) = 0 \Rightarrow T_i(0,1) \neq 2$. Indeed, if agent i chooses 0 under no incentives $T_i(0,0) = 0$, the agent will not choose 2 under $(0,1)$ since choice 2 is not incentivized. The second restriction is $T_i(0,0) = 0 \Rightarrow T_i(1,0) \neq 4$ and follows a symmetric rationale. The choice restrictions generated by these twelve blocks are listed in Table A.5.

Table A.5: Choice Restrictions of Example **E.5**

1	$T_i(0,0) = 0$	$\Rightarrow$	$T_i(0,1) \neq 2$, and $T_i(1,0) \neq 4$
2	$T_i(0,0) = 2$	$\Rightarrow$	$T_i(0,1) \neq 0$, and $T_i(1,0) = T_i(1,1) = 2$
3	$T_i(0,0) = 4$	$\Rightarrow$	$T_i(1,0) \neq 0$, and $T_i(0,1) = T_i(1,1) = 4$
4	$T_i(0,1) = 0$	$\Rightarrow$	$T_i(0,0) = 0$, $T_i(1,0) \neq 4$, and $T_i(1,1) \neq 4$,
5	$T_i(0,1) = 2$	$\Rightarrow$	$T_i(0,0) = T_i(1,0) = T_i(1,1) = 2$
6	$T_i(0,1) = 4$	$\Rightarrow$	$T_i(1,1) \neq 0$
7	$T_i(1,0) = 0$	$\Rightarrow$	$T_i(0,0) = 0$, $T_i(0,1) \neq 2$, and $T_i(1,1) \neq 2$,
8	$T_i(1,0) = 2$	$\Rightarrow$	$T_i(1,1) \neq 0$
9	$T_i(1,0) = 4$	$\Rightarrow$	$T_i(0,0) = T_i(1,0) = T_i(1,1) = 4$
10	$T_i(1,1) = 0$	$\Rightarrow$	$T_i(0,0) = T_i(0,1) = T_i(1,1) = 0$
11	$T_i(1,1) = 2$	$\Rightarrow$	$T_i(1,0) = 2$, and $T_i(0,0) \neq 4$
12	$T_i(1,1) = 4$	$\Rightarrow$	$T_i(0,1) = 4$, and $T_i(0,0) \neq 2$

These choice restrictions enable us to eliminate 72 out of the 81 possible response types. The resulting response matrix is:

$$\begin{array}{cccccccccccc} & \boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 & \boldsymbol{s}_8 & \boldsymbol{s}_9 & \\ \boldsymbol{R} = & \left[\begin{matrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 4 \\ 0 & 0 & 4 & 4 & 4 & 2 & 4 & 4 & 4 \\ 0 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 \\ 0 & 2 & 4 & 2 & 4 & 2 & 2 & 4 & 4 \end{matrix}\right. & & & & & & & & \left.\right] & \begin{matrix} T(0,0) \\ T(0,1) \\ T(1,0) \\ T(1,1) \end{matrix} \end{array}$$

It is worth noting that there is considerable overlapping among the choice restrictions. For instance, we would obtain the same response matrix if we if we were to exclude restrictions 1,3,7,and 8, correspond to the choices $T_i(0,0) = 0$, $T_i(0,1) = 0$, $T_i(1,0) = 0$, and $T_i(1,1) = 0$. Indeed, the response matrix above is also generated by choice restrictions 1,2,3,4,6,7, and 8. This subset of seven choice restrictions is displayed below:

1	$T_i(0,0) = 0$	$\Rightarrow$	$T_i(0,1) \neq 2$, and $T_i(1,0) \neq 4$
2	$T_i(0,0) = 2$	$\Rightarrow$	$T_i(0,1) \neq 0$, and $T_i(1,0) = T_i(1,1) = 2$
3	$T_i(0,0) = 4$	$\Rightarrow$	$T_i(1,0) \neq 0$, and $T_i(0,1) = T_i(1,1) = 4$
4	$T_i(0,1) = 0$	$\Rightarrow$	$T_i(0,0) = 0$, $T_i(1,0) \neq 4$, and $T_i(1,1) \neq 4$,
6	$T_i(0,1) = 4$	$\Rightarrow$	$T_i(1,1) \neq 0$
7	$T_i(1,0) = 0$	$\Rightarrow$	$T_i(0,0) = 0$, $T_i(0,1) \neq 2$, and $T_i(1,1) \neq 2$,
8	$T_i(1,0) = 2$	$\Rightarrow$	$T_i(1,1) \neq 0$

A.6 Causal Interpretation of Angrist and Imbens (1995) Monotonicity

The main paper shows that response matrix above satisfies the monotonicity condition of Angrist and Imbens (1995). A celebrated result of Angrist and Imbens (1995) is that the monotonicity condition delivers a causal interpretation to standard 2SLS estimates. The LATE parameter that compares two IV-values z, z' evaluates a weighted average of the per-unit treatment effect among the compliers that change their choice as the instrument shifts from z to z'.

The general formula for the LATE parameter that compares any two IV-values z, z' where $T_i(z) \leq T_i(z')$ is:

$$LATE(z,z') = \frac{E(Y|Z=z') - E(Y|Z=z)}{E(T|Z=z') - E(T|Z=z)} = \sum_{t<t'} E(Y(t') - Y(t)|\boldsymbol{S} \in \mathcal{S}_{t'}(z') \cap \mathcal{S}_t(z))\omega_{t,t'},$$

$$\text{where } \omega_{t,t'} = \frac{P(\boldsymbol{S} \in \mathcal{S}_{t'}(z') \cap \mathcal{S}_t(z))}{\sum_{t<t'}(t'-t) \cdot P(\boldsymbol{S} \in \mathcal{S}_{t'}(z') \cap \mathcal{S}_t(z))}, \text{ and } \mathcal{S}_t(z) = \{\boldsymbol{s} \in \mathcal{S}; \boldsymbol{s}[z] = t\}.$$

The set $\mathcal{S}_t(z)$ comprise the response-types that takes value t when the instrument is set to z. Thus $\mathcal{S}_{t'}(z') \cap \mathcal{S}_t(z)$ is the set of response types that take value t under z and t' under z'. The weights $\omega_{t,t'}$ are positive, but do not necessarily sum to one.

The LATE parameter corresponding to IV-values z_0, z_1 in the choice model given by response matrix (23) is:

$$\begin{aligned} LATE(z_1,z_0) &\equiv \frac{E(Y|Z=z_0) - E(Y|Z=z_1)}{E(T|Z=z_0) - E(T|Z=z_1)} \\ &= \frac{E(Y(t_2) - Y(t_1)|\boldsymbol{S} \in \{\boldsymbol{s}_4, \boldsymbol{s}_6\})P(\boldsymbol{S} \in \{\boldsymbol{s}_4, \boldsymbol{s}_6\}) + E(Y(t_3) - Y(t_1)|\boldsymbol{S} = \boldsymbol{s}_8)P(\boldsymbol{S} = \boldsymbol{s}_8)}{(t_2 - t_1) \cdot P(\boldsymbol{S} \in \{\boldsymbol{s}_4, \boldsymbol{s}_6\}) + (t_3 - t_1) \cdot P(\boldsymbol{S} = \boldsymbol{s}_8)}. \end{aligned}$$

If the treatment were to represent schooling years, then the LATE parameter can be interpreted as a weighted average of the causal effect of one additional year of education on the response types $(\boldsymbol{s}_4, \boldsymbol{s}_6, \boldsymbol{s}_8)$, which comprise the agents who alter their schooling choice as the instrument shifts.

A.7 Proof of Theorem T.2

The main statement of the theorem is that, under the choice rule (9), supermodular incentives imply and is implied by OMC.

OMC states that there exists an ordered sequence of treatment status $t_1 < \cdots < t_{N_T}$ in $\mathcal{T}$ and a sequence of IV-values $z_1, \ldots, z_{N_Z}$ in $\mathcal{Z}$ such that $T_i(z_1) \leq \cdots \leq T_i(z_{N_Z})$ holds for each $i \in \mathcal{I}$. It is useful to restate the condition in the following manner:

Lemma L.4. OMC holds if and only if there exists a sequence of treatment status $t_1, \ldots, t_{N_T}$ in $\mathcal{T}$ and a sequence of IV-values $z_1, \ldots, z_{N_Z}$ in $\mathcal{Z}$ such that for each $i \in \mathcal{I}$, we have that:

$$T_i(z_k) = t_j \Rightarrow T_i(z_{k+1}) \in \{t_j, t_{j+1}, ..t_{N_T}\} \text{ for all } k = 1, ..., N_Z - 1, \text{ and all } j = 1, ..., N_T - 1.$$

Proof. If OMC holds, then we have that $T_i(z_k) \leq T_i(z_{k+1}) \, \forall \, i \in \mathcal{I}$ and all $k = 1, ..., N_Z - 1$. Thus, it must be the case that if $T_i(z_k) = t_j$ then $T_i(z_k) \geq t_j$, that is $T_i(z_k) \in \{t_j, t_{j+1}, ..., t_{N_T}\}$. On the other hand, suppose that $T_i(z_k) = t_j \Rightarrow T_i(z_{k+1}) \in \{t_j, t_{j+1}, ..t_{N_T}\}$ for all $k = 1, ..., N_Z - 1$. Thus, consider assigning values to the treatment choices such that $t_1 < ... < t_{N_T}$. This case implies the choice restriction $T_i(z_k) = t_j \Rightarrow T_i(z_{k+1}) \geq t_j$ for all $k = 1, ..., N_Z - 1$. If this constraint applies to any $j \in \{1, ..., N_T\}$, then we have that $T_i(z_k) \leq T_i(z_{k+1})$ for all $k = 1, ..., N_Z - 1$, which completes the proof. □

We now prove that supermodular incentives imply OMC given the choice rule (9).

Proof. Consider a sequence of IV-values $z_1, ..., z_{N_Z}$ and a sequence of treatment choices $t_1, ...t_{N_T}$ for which supermodularity holds. We first examine the choices t_{j+1} versus t_j for an IV-change from z_k to z_{k+1}. Let $T_i(z_k) = t_{j+1}$, under supermodular incentives, we have that $\boldsymbol{L}[z_{k+1}, t_j] - \boldsymbol{L}[z_k, t_j] \leq \boldsymbol{L}[z_{k+1}, t_{j+1}] - \boldsymbol{L}[z_k, t_{j+1}]$. Thus, according to the choice rule (9), $T_i(z_k) \neq t_j$. In summary, we have that $T_i(z_k) = t_{j+1} \Rightarrow T_i(z_{k+1}) \neq t_j$. We can extend this rationale to compare choice t_{j+1} versus t_α for $\alpha = 1, ..., j$. Note that $\boldsymbol{L}[z_{k+1}, t_\alpha] - \boldsymbol{L}[z_k, t_\alpha] \leq \boldsymbol{L}[z_{k+1}, t_{j+1}] - \boldsymbol{L}[z_k, t_{j+1}]$ for $\alpha = 1, ..., j$. Thus, we have that $T_i(z_k) = t_{j+1} \Rightarrow T_i(z_{k+1}) \neq t_\alpha$ for all $\alpha \in \{1, ..., j\}$. This means that $T_i(z_k) = t_{j+1} \Rightarrow T_i(z_{k+1}) \notin \{t_1, ..., t_j\}$ for all $j = 1, ..., N_T - 1$. Otherwise stated, we have that $T_i(z_k) = t_{j+1} \Rightarrow T_i(z_{k+1}) \in \{t_{j+1}, ..., t_{N_T}\}$ for all $j = 1, ..., N_T - 1$. According to Lemma **L.4**, OMC holds. □

Next we show that under the choice rule (9), OMC implies supermodular incentives.

Proof. According to Lemma **L.4**, if OMC holds, then there must exist a sequence of treatment status $t_1, \ldots, t_{N_T}$ in $\mathcal{T}$ and a sequence of IV-values $z_1, \ldots, z_{N_Z}$ in $\mathcal{Z}$ such that $T_i(z_k) = t_j \Rightarrow T_i(z_{k+1}) \in \{t_j, t_{j+1}, ..t_{N_T}\}$ for all $k = 1, ..., N_Z - 1$, all $j = 1, ..., N_T - 1$, and all $i \in \mathcal{I}$. Thus we must have that $T_i(z_k) = t_j \Rightarrow T_i(z_{k+1}) \neq t_\alpha$ for all $\alpha \in \{1, ..., j-1\}$. According to the choice rule (9), we must have that $\boldsymbol{L}[z_{k+1}, t_j] - \boldsymbol{L}[z_k, t_j] \geq \max_{\alpha=1}^{j-1} \boldsymbol{L}[z_{k+1}, t_\alpha] - \boldsymbol{L}[z_k, t_\alpha]$. But this property applies to all $j = 2, ..., N_T$. Thus we must have that $\boldsymbol{L}[z_{k+1}, t_j] - \boldsymbol{L}[z_k, t_j] \geq \boldsymbol{L}[z_{k+1}, t_{j-1}] - \boldsymbol{L}[z_k, t_{j-1}]$ for all $j = 2, ..., N_T$. Otherwise stated, we must have that $\boldsymbol{L}[z_{k+1}, t_{j-1}] - \boldsymbol{L}[z_k, t_{j-1}] \leq \boldsymbol{L}[z_{k+1}, t_{j-1}] - \boldsymbol{L}[z_k, t_{j-1}]$ for all $j = 2, ..., N_T$. This property holds for any $k \in \{1, ..., N_Z - 1\}$, which is the definition of OMC. □

A.8 Proof of Theorem T.3

We seek to prove that, if the incentive matrix $\boldsymbol{L}$ is binary, then monotonic incentives imply UMC. Recall that UMC holds if and only if no 2×2 submatrix in $\boldsymbol{R}$ that exhibits the prohibit pattern which displays a choice t in one of its diagonals while displays no t in the other diagonal. Specifically, no 2×2 submatrix in $\boldsymbol{R}$ can take the form:

$$\begin{array}{cc} \begin{array}{cc} \boldsymbol{s} & \boldsymbol{s}' \end{array} & \\ \begin{bmatrix} t & t'' \\ t' & t \end{bmatrix} & \begin{array}{c} T_i(z) \\ T_i(z') \end{array} \end{array}, \tag{72}$$

where $t, t', t'' \in \mathcal{T}$ and $z, z' \in \mathcal{Z}$. It is useful to investigate how the prohibit patter (72) in light of the choice rule (9). Let $\boldsymbol{L}'$ be the 2×3 submatrix of the binary incentive matrix $\boldsymbol{L}$ corresponding to rows z, z' and columns t, t', t''.

Consider the first type in (72) $\boldsymbol{s} = [t, t']'$. For $\boldsymbol{s}$ to arise it must be the case that $T_i(z) = t \nRightarrow T_i(z') \neq t'$ According to the choice rule (9), this lack of choice restriction can only arise when:[32]

$$\boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] > \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t]. \tag{73}$$

Given that the incentive matrix is binary, this must the be case that:

1. $\boldsymbol{L}[z', t'] > \boldsymbol{L}[z, t']$ and $\boldsymbol{L}[z', t] \leq \boldsymbol{L}[z, t]$; or
2. $\boldsymbol{L}[z', t'] \geq \boldsymbol{L}[z, t']$ and $\boldsymbol{L}[z', t] < \boldsymbol{L}[z, t]$

Now consider the second type in (72) $\boldsymbol{s}; = [t'', t]'$. For $\boldsymbol{s}'$ to arise it must be the case that $T_i(z') = t \nRightarrow T_i(z) \neq t''$. This lack of choice restriction can only arise when

$$\boldsymbol{L}[z, t''] - \boldsymbol{L}[z', t''] > \boldsymbol{L}[z, t] - \boldsymbol{L}[z', t]. \tag{74}$$

Given that the incentive matrix is binary, this must the be case that:

1. $\boldsymbol{L}[z, t''] > \boldsymbol{L}[z', t'']$ and $\boldsymbol{L}[z, t] \leq \boldsymbol{L}[z', t]$; or
2. $\boldsymbol{L}[z, t''] = \boldsymbol{L}[z', t'']$ and $\boldsymbol{L}[z, t] < \boldsymbol{L}[z', t]$

It is clear that the only possibility to generate the prohibit parte in by combining the first item of the two lists, namely,

$$\boldsymbol{L}[z', t'] > \boldsymbol{L}[z, t'], \quad \boldsymbol{L}[z, t''] > \boldsymbol{L}[z', t''], \text{ and } \boldsymbol{L}[z, t] = \boldsymbol{L}[z', t].$$

In other words, the prohibit pattern requires the following pattern of incentives:

1. Incentives for t must be equal $\boldsymbol{L}[z, t] = \boldsymbol{L}[z', t]$
2. Incentives for t' must increase as Z changes from z to z' : $\boldsymbol{L}[z, t'] < \boldsymbol{L}[z', t']$.
3. Incentives for t'' must decrease as Z changes from z to z' : $\boldsymbol{L}[z, t''] > \boldsymbol{L}[z', t'']$.

The pattern of incentives for t and t' violate the monotonic incentive condition, which proves the theorem.

[32] Alternatively, one can state that the type only arises when $T_i(z') = t' \nRightarrow T_i(z) \neq t$. According to the choice rule (9), this lack of choice restriction can only arise when $\boldsymbol{L}[z, t] - \boldsymbol{L}[z', t] > \boldsymbol{L}[z, t'] - \boldsymbol{L}[z', t']$. It turns out that the incentive relationship above is equivalent to the incentive relationship in (74).

A.9 Proof of Theorem T.4

We first seek to prove that t-monotonic incentives (33) implies the monotonicity condition (32). To do so, it suffices to prove that t-monotonic incentives prevents the advent of the prohibit pattern in the response matrix $\boldsymbol{R}$. Specifically, no 2×2 submatrix in $\boldsymbol{R}$ can take the form:

$$\begin{array}{cc} \boldsymbol{s} & \boldsymbol{s}' \end{array} \\ \begin{bmatrix} t & t'' \\ t' & t \end{bmatrix} \begin{array}{l} T_i(z) \\ T_i(z') \end{array}, \tag{75}$$

where $t, t', t'' \in \mathcal{T}$ and $z, z' \in \mathcal{Z}$. Consider the IV-values $z, z' \in \mathcal{Z}$. If t-monotonic incentives hold, there are two cases to consider.

The first case consists the instance where

$$\boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \leq \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \, \forall \, t' \in \mathcal{T} \setminus \{t\}$$

holds. According to the choice rule (9), it must be the case that $T_i(z) = t \Rightarrow T_i(z') \neq t' \forall t' \in \mathcal{T} \setminus \{t\}$, which is equivalent to state that $T_i(z) = t \Rightarrow T_i(z') = t$ which prevents the prohibit pattern.

The second case is where the following condition holds:

$$\boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] \geq \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \, \forall \, t' \in \mathcal{T} \setminus \{t\}$$

This condition can be equivalently stated as:

$$\boldsymbol{L}[z, t] - \boldsymbol{L}[z', t] \leq \boldsymbol{L}[z, t'] - \boldsymbol{L}[z', t'] \, \forall \, t' \in \mathcal{T} \setminus \{t\}.$$

Applying the same rationale of the first case, we have that $T_i(z') = t \Rightarrow T_i(z) = t$ which also prevents the prohibit pattern.

Next we seek to prove that if the monotonicity condition (32) holds, than t-monotonic incentives (33) must be satisfied. For the monotonicity condition (32) to hold, the prohibit pattern (75) cannot occur. The prohibit pattern requires two conditions to occur:

1. $T_i(z) = t$ must not imply $T_i(z') = t'$ for some $t' \in \mathcal{T} \setminus \{t\}$; and
2. $T_i(z') = t$ must not imply $T_i(z) = t''$ for some $t'' \in \mathcal{T} \setminus \{t\}$.

According to the choice rule (9), these two conditions require the following incentive relationships:

1. $\boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] < \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t']$ for some $t' \in \mathcal{T} \setminus \{t\}$, and
2. $\boldsymbol{L}[z, t] - \boldsymbol{L}[z', t] < \boldsymbol{L}[z, t''] - \boldsymbol{L}[z', t'']$ for some $t'' \in \mathcal{T} \setminus \{t\}$.

These conditions imply that the prohibit patters requires the following incentive scheme:

$$\boldsymbol{L}[z', t''] - \boldsymbol{L}[z, t''] < \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] < \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \text{ for some } t', t'' \in \mathcal{T} \setminus \{t\}.$$

Otherwise stated, the prohibit pattern requires that the incentive difference for choice t be strictly larger than the minimum difference among the choices and strictly smaller than the maximum difference among the treatment choices. Consequently if the prohibit pattern does not occur, then it must be the case that:

$$\boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] = \max_{t' \in \mathcal{T}} \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'] \quad \text{or} \quad \boldsymbol{L}[z', t] - \boldsymbol{L}[z, t] = \min_{t' \in \mathcal{T}} \boldsymbol{L}[z', t'] - \boldsymbol{L}[z, t'].$$

This condition is equivalent to t-monotonic incentives (33).

A.10 Examples of Incentive IV Models where UMC Holds

Equations (76)–(77) display the incentive matrices in (35)–(36) and the corresponding response matrices generated by the applying the Choice Rule (9) to each of the incentive matrices.

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array}, \quad \boldsymbol{R} = \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{s}_2 & \boldsymbol{s}_1 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \boldsymbol{s}_6 & \boldsymbol{s}_7 \end{array} \\ \begin{bmatrix} t_1 & t_1 & t_1 & t_2 & t_2 & t_3 & t_3 \\ t_1 & t_1 & t_1 & t_1 & t_2 & t_1 & t_3 \\ t_1 & t_2 & t_3 & t_2 & t_2 & t_3 & t_3 \end{bmatrix} \end{array} \begin{array}{c} \\ T(z_1) \\ T(z_2) \\ T(z_3) \end{array} \tag{76}$$

$$\boldsymbol{L} = \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array}, \quad \boldsymbol{R} = \begin{array}{c} \begin{array}{ccccc} \boldsymbol{s}_2 & \boldsymbol{s}_1 & \boldsymbol{s}_3 & \boldsymbol{s}_4 & \boldsymbol{s}_5 \end{array} \\ \begin{bmatrix} t_1 & t_1 & t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 & t_2 & t_3 \\ t_1 & t_2 & t_3 & t_2 & t_3 \end{bmatrix} \end{array} \begin{array}{c} \\ T(z_1) \\ T(z_2) \\ T(z_3) \end{array} \tag{77}$$

The two response matrices above adhere to the UMC since there are no 2×2 submatrix that contains a choice $t \in \{t_1, t_2, t_3\}$ in the diagonal, but does not contain the same choice t in its off-diagonal.

A.11 Proof of Theorem C.3

We seek to show that for any binary incentive matrix $\boldsymbol{L}$, t-Monotonic Incentives (33) holds for a choice t if and only if matrices $\boldsymbol{L}_t^1$ and $\boldsymbol{L}_t^0$ are lonesum. In this notation $\boldsymbol{L}_t^1$ is the submatrix consistent of the z-rows in $\boldsymbol{L}$ such that $\boldsymbol{L}[z,t] = 1$. On the other hand, $\boldsymbol{L}_t^0$ consistent of the z-rows in $\boldsymbol{L}$ such that $\boldsymbol{L}[z,t] = 0$. It is useful to define the following sets of IV-values $\mathcal{Z}_1 = \{z \in \mathcal{Z}; \boldsymbol{L}[z,t] = 1\}$ and $\mathcal{Z}_0 = \{z \in \mathcal{Z}; \boldsymbol{L}[z,t] = 0\}$.

We first show that if $\boldsymbol{L}_t^1$ and $\boldsymbol{L}_t^0$ are lonesum, then t-Monotonic Incentives holds. For any two IV-values $z \in \mathcal{Z}_0$ and $z' \in \mathcal{Z}_1$ we have that $\boldsymbol{L}[z',t] - \boldsymbol{L}[z,t] \geq \boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \,\forall\, t' \in \mathcal{T} \setminus \{t\}$ since $\boldsymbol{L}[z',t] - \boldsymbol{L}[z,t] = 1$. For any two IV-values $z, z' \in \mathcal{Z}_0$ we have that $\boldsymbol{L}[z',t] - \boldsymbol{L}[z,t] = 0$. But $\boldsymbol{L}_t^0$ is lonesum, thus $\boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \geq 0 \,\forall\, t' \in \mathcal{T} \setminus \{t\}$ or $\boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \leq 0 \,\forall\, t' \in \mathcal{T} \setminus \{t\}$. Thus, it must me the case that $\boldsymbol{L}[z',t] - \boldsymbol{L}[z,t] \geq \boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \,\forall\, t' \in \mathcal{T} \setminus \{t\}$ or $\boldsymbol{L}[z',t] - \boldsymbol{L}[z,t] \leq \boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] \,\forall\, t' \in \mathcal{T} \setminus \{t\}$. The symmetric argument applies to any two IV-values $z, z' \in \mathcal{Z}_1$.

We now prove that if $\boldsymbol{L}$ is binary and t-Monotonic Incentives holds, then it must be the case that $\boldsymbol{L}_t^1$ and $\boldsymbol{L}_t^0$ are lonesum. Recall that t-Monotonic Incentives states that $\boldsymbol{L}[z,t] - \boldsymbol{L}[z',t] \leq \boldsymbol{L}[z',t'] - \boldsymbol{L}_t^0[z',t']$ for all $t' \in \mathcal{T} \setminus \{t\}$ or $\boldsymbol{L}[z,t] - \boldsymbol{L}[z',t] \geq \boldsymbol{L}[z',t'] - \boldsymbol{L}_t^0[z',t']$ for all $t' \in \mathcal{T} \setminus \{t\}$. Note also that for any two IV-values $z, z' \in \mathcal{Z}_0$, we have that $\boldsymbol{L}_t^0[z,t] = \boldsymbol{L}_t^0[z',t] = 0$. Thus, $\boldsymbol{L}[z,t] - \boldsymbol{L}[z',t] = 0$.

Now suppose that $\boldsymbol{L}_t^0$ is not lonesum. Thus there must exist a 2×2 submatrix in $\boldsymbol{L}_t^0$ that displays the prohibit pattern of an identity matrix. This means that there must exist $z, z' in \mathcal{Z}_0$ and two treatment values $\tilde{t}, \tilde{t}'$ such that $\boldsymbol{L}[z,\tilde{t}] = \boldsymbol{L}[z',\tilde{t}'] = 1$ while $\boldsymbol{L}[z',\tilde{t}] = \boldsymbol{L}[z,\tilde{t}'] = 0$. In this case, we have that $\boldsymbol{L}[z',\tilde{t}] - \boldsymbol{L}_t^0[z',\tilde{t}] = -1$ while $\boldsymbol{L}[z',\tilde{t}'] - \boldsymbol{L}_t^0[z',\tilde{t}'] = 1$. This violates the

t-Monotonic Incentives criteria since: $\boldsymbol{L}[z,t]-\boldsymbol{L}[z',t]<\boldsymbol{L}[z',\tilde{t}']-\boldsymbol{L}_t^0[z',\tilde{t}']$ while $\boldsymbol{L}[z,t]-\boldsymbol{L}[z',t]>\boldsymbol{L}[z',\tilde{t}]-\boldsymbol{L}_t^0[z',\tilde{t}]$. The symmetric proof applies for the case of $\boldsymbol{L}_t^1$.

A.12 Proof of Theorem T.5

we seek to prove that t-EMCO holds if and only if t-CIG is satisfied. Recall that the theorem applies to IV models described by Assumptions (1)–(3) whose choice incentives are determined by an incentive matrix $\boldsymbol{L}$ that satisfies Choice Rule (9).

We first prove that t-CIG implies t-EMCO. Consider the IV-change from z to z'. Let $\boldsymbol{L}[z,t']-\boldsymbol{L}[z',t']=c'$ for some $t'\in\mathcal{T}\setminus\{t\}$. The t-CIG condition (43) states that $\boldsymbol{L}[z,t']-\boldsymbol{L}[z',t']=c'$ for all $t'\in\mathcal{T}\setminus\{t\}$. Applying the Choice Rule (9) to $T_i(z)=t'$, we have that:

$$T_i(z)=t'\Rightarrow T_i(z')\neq t'' \text{ for all } t''\in\mathcal{T}\setminus\{t,t'\}. \tag{78}$$

Now let $\boldsymbol{L}[z,t]-\boldsymbol{L}[z',t]=c$. We have two possibilities: $c\leq c'$ or $c>c'$. If $c\leq c'$, we can apply the choice rule and obtain the $T_i(z)=t'\Rightarrow T_i(z)\neq t$, which, when combined with result (78), implies that $T_i(z)=t'\Rightarrow T_i(z')=t'$. On the other hand, applying the choice rule to $T_i(z)=t$ generates **no** restriction. However, applying the choice rule to $T_i(z')=t$ we obtain that $T_i(z)=t\Rightarrow T_i(z)\neq t'$ for all $t'\in\mathcal{T}\setminus\{t\}$. Otherwise stated, we have that $T_i(z')=t\Rightarrow T_i(z)=t$. These results satisfy the following t-EMCO inequalities:

$$\mathbf{1}[T_i(z)=t]\geq\mathbf{1}[T_i(z')=t]\forall\, i \text{ and } \mathbf{1}[T_i(z)=t']\leq\mathbf{1}[T_i(z')=t'];\forall\, i\in\mathcal{I},t'\in\mathcal{T}\setminus\{t\}.$$

Now suppose $c>c'$. Applying the choice rule to compare t' and t does **not** generate the restriction that $T_i(z)=t'\Rightarrow T_i(z')\neq t$. Combining this fact with result (78) we have that $T_i(z)=t'\Rightarrow T_i(z')\in\{t,t'\}$. On the other hand, if we apply the choice rule to $T_i(z)=t$ against any choice $t'\in\mathcal{T}\setminus\{t\}$ generates the choice restriction $T_i(z)=t\Rightarrow T_i(z)\neq t'$ for any $t'\in\mathcal{T}\setminus\{t\}$. Thus, we that $T_i(z)=t\Rightarrow T_i(z)=t$. These results satisfy the following inequality condition:

$$\mathbf{1}[T_i(z)=t]\leq\mathbf{1}[T_i(z')=t]\forall\, i \text{ and } \mathbf{1}[T_i(z)=t']\geq\mathbf{1}[T_i(z')=t'];\forall\, i\in\mathcal{I},t'\in\mathcal{T}\setminus\{t\}.$$

This proves that t-CIG implies t-EMCO.

We now seek to prove that t-EMCO implies t-CIG. To do so, note that the monotonicity condition $\mathbf{1}[T_i(z)=t']\leq\mathbf{1}[T_i(z')=t']$ is equivalent to the choice restriction $T_i(z)=t'\Rightarrow T_i(z')=t'$, which implies that $T_i(z')\neq t''$, for any $t''\in\mathcal{T}\setminus\{t'\}$. According to the choice rule, we must have that $\boldsymbol{L}[z',t'']-\boldsymbol{L}[z,t'']\leq\boldsymbol{L}[z',t']-\boldsymbol{L}[z,t']$ for all $t''\in\mathcal{T}\setminus\{t'\}$. In summary, we have that:

$$\mathbf{1}[T_i(z)=t']\leq\mathbf{1}[T_i(z')=t']\forall\, i\Rightarrow\boldsymbol{L}[z',t'']-\boldsymbol{L}[z,t'']\leq\boldsymbol{L}[z',t']-\boldsymbol{L}[z,t'] \text{ for all } t''\in\mathcal{T}\setminus\{t'\}. \tag{79}$$

The t-EMCO condition is defined in (38)–(39). We first focus on equation (38), which states that for all $t'\in\mathcal{T}\setminus\{t\}$, we have that $\mathbf{1}[T_i(z)=t']\geq\mathbf{1}[T_i(z')=t'];\forall\, i\in\mathcal{I}$. Consider t' and t'' in $\mathcal{T}\setminus\{t\}$. According to (79), $\mathbf{1}[T_i(z)=t']\geq\mathbf{1}[T_i(z')=t']$ implies that $\boldsymbol{L}[z',t'']-\boldsymbol{L}[z,t'']\leq\boldsymbol{L}[z',t']-\boldsymbol{L}[z,t']$. On the other hand, $\mathbf{1}[T_i(z)=t'']\geq\mathbf{1}[T_i(z')=t'']$ implies that $\boldsymbol{L}[z',t']-\boldsymbol{L}[z,t']\leq\boldsymbol{L}[z',t'']-\boldsymbol{L}[z,t'']$.

Thus, it must be the case that $\boldsymbol{L}[z',t'] - \boldsymbol{L}[z,t'] = \boldsymbol{L}[z',t''] - \boldsymbol{L}[z,t'']$. This rationale applies to all $t' \in \mathcal{T} \setminus \{t\}$. Equation (39) states that for all $t' \in \mathcal{T} \setminus \{t\}$, we have that $\mathbf{1}[T_i(z) = t'] \leq \mathbf{1}[T_i(z') = t']; \forall i \in \mathcal{I}$. We can apply the symmetric rationale to obtain that $\boldsymbol{L}[z,t'] - \boldsymbol{L}[z',t'] = \boldsymbol{L}[z,t''] - \boldsymbol{L}[z',t'']$. We then conclude that t-EMCO implies t-CIG.

A.13 Proof of Corollary C.4

Let $\boldsymbol{L}$ be an incentive matrix where t-CIG (43) holds. The generated response matrix is saturated w.r.t. to t-EMCO (38) if and only if $\boldsymbol{L}[z,t] - \boldsymbol{L}[z',t] \neq \boldsymbol{L}[z,t'] - \boldsymbol{L}[z',t']$ for $t' \neq t$ and all $z,z' \in \mathcal{Z}$ such that $z \neq z'$.

According to **T.5**, t-CIG (43) implies t-EMCO (38)–(39). Without loss of generality, we can assume that for $z,z' \in \mathcal{Z}$, we have that:

$$\mathbf{1}[T_i(z) = t] \leq \mathbf{1}[T_i(z') = t] \forall\, i \text{ and } \mathbf{1}[T_i(z) = t'] \geq \mathbf{1}[T_i(z') = t']; \forall\, i \in \mathcal{I}, t' \in \mathcal{T} \setminus \{t\}.$$

Recall that the monotonicity inequality

A.14 Doubly Robust Estimation Algorithm for Response-type Probabilities

Step 1. Partition the sample index $\mathcal{I} = \{1,\ldots,n\}$ into K subsets such that $\cup_{k=1}^{K}\{\mathcal{I}_k\} = \mathcal{I}$, where the number of partitions K is commonly fixed to five. Let $\mathcal{I}_k^c = \mathcal{I} \setminus \mathcal{I}_k$ be the complement of $\mathcal{I}_k$

Step 2. For each value $t \in \{1,2,3\}$ and each partition k, compute the estimator $\hat{\gamma}_{t,k,\boldsymbol{s}}$ associated with the kappa function $\kappa_{\boldsymbol{s}}(t,Z,X)$ by minimizing the following expression:

$$\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k} \in \arg\min_{\boldsymbol{\gamma} \in \mathbf{R}^p} \sum_{i \in \mathcal{I}_k^c} \left(\frac{1}{2} (\boldsymbol{h}(Z_i,X_i)'\boldsymbol{\gamma})^2 + \sum_{z \in \mathcal{Z}} \nu_{\boldsymbol{s}}(t,z) \boldsymbol{h}(z,X_i)'\boldsymbol{\gamma} \right) + \alpha_{\boldsymbol{\gamma}} \|\boldsymbol{\gamma}\|_1, \tag{80}$$

where $\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}$ is evaluated using all data that is not in $\mathcal{I}_k$, while $\alpha_{\boldsymbol{\gamma}}$ is the penalty parameter determined by a cross-validation procedure employing all sampling data.

Step 3. For each value $t \in \{1,2,3\}$ and each partition I_k, compute the estimator $\hat{\boldsymbol{\beta}}_{t,k}$ associated with the propensity score $P(T = t|Z,X)$ via the least absolute shrinkage and selection operator (lasso) procedure that minimizes the following expression:

$$\hat{\boldsymbol{\beta}}_{t,k} \in \arg\min_{\boldsymbol{\beta} \in \mathbf{R}^p} \sum_{i \in \mathcal{I}_k^c} (1[T_i = t] - \boldsymbol{h}(Z_i,X_i)'\boldsymbol{\beta})^2 + \alpha_{\boldsymbol{\beta}} \|\boldsymbol{\beta}\|_1,$$

where $\alpha_{\boldsymbol{\beta}}$ is the penalty parameter also determined by via cross-validation procedure.[33]

Step 4. Given $\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}$ and $\hat{\boldsymbol{\beta}}_{t,k}$, we compute the orthogonal score estimator $\hat{\psi}_{\boldsymbol{s},i,k}$ for each participant

[33] Note that the penalty parameters $\alpha_{\boldsymbol{\beta}}$ and $\alpha_{\boldsymbol{\gamma}}$ do not need to be the same, but the functions $\boldsymbol{h}(Z,X)$ are the same in steps 2 and 3.

$i \in \mathcal{I}_k$ and for each partition k :

$$\hat{\psi}_{\boldsymbol{s},k,i} \equiv \sum_{t\in\mathcal{T}} \left(\boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k} \cdot \left(1[T_i = t] - \boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\beta}}_{t,k}\right) + \sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(t, z)\boldsymbol{h}(z, X_i)'\hat{\boldsymbol{\beta}}_{t,k} \right).$$

Step 5. The estimator for the propensity score $P(\boldsymbol{S} = \boldsymbol{s})$ is the average of the orthogonal scores within partition, that is, $\hat{\psi}_{\boldsymbol{s},k} = |\mathcal{I}_k|^{-1} \sum_{i\in\mathcal{I}_k} \sum_{t\in\mathcal{T}} \hat{\psi}_{\boldsymbol{s},k,i}$. The final estimate is the average of the orthogonal scores across partitions, namely, $\hat{\psi}_{\boldsymbol{s}} = n^{-1} \sum_{k=1}^{K} \hat{\psi}_{\boldsymbol{s},k} \cdot |\mathcal{I}_k|$.

Step 6. Inference is performed via the bootstrap multiplier method. For each partition k, we draw B samples $\{W_i^{(b)}\}_{i\in\mathcal{I}_k}$ of i.i.d. standard normals to compute:

$$\hat{\psi}_{\boldsymbol{s},k}^{(b)} = \hat{\psi}_{\boldsymbol{s},k} + \frac{1}{n}\sum_{i\in\mathcal{I}_k}^{n} W_i^{(b)}\left(\hat{\psi}_{\boldsymbol{s},k,i} - \hat{\psi}_{\boldsymbol{s},k}\right), \text{ and } \hat{\psi}_{\boldsymbol{s}}^{(b)} = n^{-1}\sum_{k=1}^{K} \hat{\psi}_{\boldsymbol{s},k}^{(b)} \cdot |\mathcal{I}_k|.$$

We use the distribution of $\hat{\psi}_{\boldsymbol{s}}^{(b)}$ to compute the standard error of the estimator for the type probability.

A few notes on the estimation method are in order. The sample splitting in Step 1 is not necessary to secure normality of the estimator and can be voided. The estimators in Steps 2 and 3 allow for some degree of flexibility. In our setup, $(Z, X)'\hat{\boldsymbol{\beta}}_{t,k}$ estimates the propensity score and $\boldsymbol{h}(Z, X)'\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}$ estimates the kappa function. These estimates can be obtained by suitable alternative machine learning estimators. For instance, it is possible to transform the minimization that evaluates $\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}$ in Step 2 into a standard lasso-type estimator.

Let $\boldsymbol{H}_k(z) \equiv \boldsymbol{h}(z, \boldsymbol{X})$ denotes the $|\mathcal{I}_k^c| \times p$ matrices that stack $\boldsymbol{h}(z, X_i)'$ across participants $i \in \mathcal{I}_k^c$. In the same token, let $\boldsymbol{H}_k \equiv \boldsymbol{h}(\boldsymbol{Z}, \boldsymbol{X})$ be the matrix that stakes $\boldsymbol{h}(Z_i, X_i)'$ across $i \in \mathcal{I}_k^c$, and let $\boldsymbol{\iota}_k$ be the $|\mathcal{I}_k^c|$-dimensional vector of ones. In this notation, the minimization of Step 2 can be equivalently expressed as:[34]

$$\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k} \in \arg\min_{\boldsymbol{\gamma}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} (\boldsymbol{h}(Z_i, X_i)'\boldsymbol{\theta} - \boldsymbol{h}(Z_i, X_i)'\boldsymbol{\gamma})^2 + \alpha_{\boldsymbol{\gamma}}\|\boldsymbol{\gamma}\|_1, \text{ where } \boldsymbol{\theta} \equiv (\boldsymbol{H}_k'\boldsymbol{H}_k)^{-1} \left(\sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(t, z)\boldsymbol{H}_k(z)'\boldsymbol{\iota}_k\right).$$

The term $\boldsymbol{h}(Z_i, X_i)'\boldsymbol{\theta}$ can be roughly understood as the projection of the function $\sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(t, z)\boldsymbol{h}(z, X_i)$ into the space generated by $\boldsymbol{h}(Z_i, X_i)$. Finally, we use the leave-one-out sampling scheme in all cross-validation methods.

A.15 Doubly Robust Estimation Algorithm for Identified Counterfactual Outcomes

Step 1. Partition $\mathcal{I}$ into $\cup_{k=1}^{K}\{\mathcal{I}_k\} = \mathcal{I}$, where $\mathcal{I}_k^c = \mathcal{I} \setminus \mathcal{I}_k$.

[34]The estimator is numerically equivalent to evaluating the minimum of the function in Step 2. The equivalence is easy be shown when expressing the minimization using matrix notation.

Step 2. For each k, compute the estimator $\hat{\gamma}_{t,k,\boldsymbol{s}}$ as:

$$\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k} \in \arg\min_{\boldsymbol{\gamma}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(\frac{1}{2}(\boldsymbol{h}(Z_i,X_i)'\boldsymbol{\gamma})^2 + \sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s},t}(z)\boldsymbol{h}(z,X_i)'\boldsymbol{\gamma} \right) + \alpha_{\boldsymbol{\gamma}}\|\boldsymbol{\gamma}\|_1, \tag{81}$$

where $\alpha_{\boldsymbol{\gamma}}$ is the penalty parameter determined by a cross-validation (leave-one-out) procedure.

Step 3. For each partition k, compute the estimators $\hat{\boldsymbol{\beta}}_{t,k}$, and $\hat{\boldsymbol{\theta}}_{t,k}$ via lasso:

$$\hat{\boldsymbol{\theta}}_{t,k} \in \arg\min_{\boldsymbol{\theta}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \big(Y\cdot 1[T_i=t] - \boldsymbol{h}(Z_i,X_i)'\boldsymbol{\theta}\big)^2 + \alpha_{\boldsymbol{\theta}}\|\boldsymbol{\theta}\|_1,$$

$$\hat{\boldsymbol{\beta}}_{t,k} \in \arg\min_{\boldsymbol{\beta}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \big(1[T_i=t] - \boldsymbol{h}(Z_i,X_i)'\boldsymbol{\beta}\big)^2 + \alpha_{\boldsymbol{\beta}}\|\boldsymbol{\beta}\|_1,$$

where $\alpha_{\boldsymbol{\beta}}, \alpha_{\boldsymbol{\theta}}$ are the penalty parameters determined by cross-validation.

Step 4. Given $\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}, \hat{\boldsymbol{\beta}}_{t,k}$, and $\hat{\boldsymbol{\theta}}_{t,k}$, for each agent $i\in\mathcal{I}_k$ and each partition k, compute the orthogonal score $\hat{\psi}_{\boldsymbol{s},i,k}$ for $P(\boldsymbol{S}=\boldsymbol{s})$ and $\hat{\varphi}_{\boldsymbol{s},i,k}$ for $E(Y\mathbf{1}[\boldsymbol{S}=\boldsymbol{s}])$

$$\hat{\psi}_{\boldsymbol{s},k,i} \equiv \left(\boldsymbol{h}(Z_i,X_i)'\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}\cdot\Big(1[T_i=t] - \boldsymbol{h}(Z_i,X_i)'\hat{\boldsymbol{\beta}}_{t,k}\Big) + \sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(t,z)\boldsymbol{h}(z,X_i)'\hat{\boldsymbol{\beta}}_{t,k} \right),$$

$$\hat{\varphi}_{\boldsymbol{s},k,i} \equiv \left(\boldsymbol{h}(Z_i,X_i)'\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}\cdot\Big(Y\cdot 1[T_i=t] - \boldsymbol{h}(Z_i,X_i)'\hat{\boldsymbol{\theta}}_{t,k}\Big) + \sum_{z\in\mathcal{Z}} \nu_{\boldsymbol{s}}(t,z)\boldsymbol{h}(z,X_i)'\hat{\boldsymbol{\theta}}_{t,k} \right).$$

Step 5. The estimator for $P(\boldsymbol{S}=\boldsymbol{s})$ is the average of the orthogonal scores $\hat{\psi}_{\boldsymbol{s}} = n^{-1}\sum_{k=1}^K \hat{\psi}_{\boldsymbol{s},k}\cdot|\mathcal{I}_k|$, where $\hat{\psi}_{\boldsymbol{s},k} = |\mathcal{I}_k|^{-1}\sum_{i\in\mathcal{I}_k}\sum_{t\in\mathcal{T}}\hat{\psi}_{\boldsymbol{s},k,i}$. The estimator for $E(Y(t)\mathbf{1}[\boldsymbol{S}=\boldsymbol{s}])$ is also the average of the orthogonal scores $\hat{\varphi}_{\boldsymbol{s}} = n^{-1}\sum_{k=1}^K \hat{\varphi}_{\boldsymbol{s},k}\cdot|\mathcal{I}_k|$, where $\hat{\varphi}_{\boldsymbol{s},k} = |\mathcal{I}_k|^{-1}\sum_{i\in\mathcal{I}_k}\sum_{t\in\mathcal{T}}\hat{\varphi}_{\boldsymbol{s},k,i}$. The final estimator for $E(Y(t)|\boldsymbol{S}=\boldsymbol{s})$ is the ratio $\hat{\varphi}_{\boldsymbol{s}}/\hat{\psi}_{\boldsymbol{s}}$.

Step 6. Our inference uses a multiplier bootstrap that draw B samples $\{W_i^{(b)}\}_{i\in\mathcal{I}_k}$ of i.i.d. standard normals for each partition k. We then compute both scores:

$$\hat{\psi}_{\boldsymbol{s},k}^{(b)} = \hat{\psi}_{\boldsymbol{s},k} + \frac{1}{n}\sum_{i\in\mathcal{I}_k}^{n} W_i^{(b)}\big(\hat{\psi}_{\boldsymbol{s},k,i} - \hat{\psi}_{\boldsymbol{s},k}\big), \text{ and } \hat{\psi}_{\boldsymbol{s}}^{(b)} = n^{-1}\sum_{k=1}^{K}\hat{\psi}_{\boldsymbol{s},k}^{(b)}\cdot|\mathcal{I}_k|,$$

$$\hat{\varphi}_{\boldsymbol{s},k}^{(b)} = \hat{\varphi}_{\boldsymbol{s},k} + \frac{1}{n}\sum_{i\in\mathcal{I}_k}^{n} W_i^{(b)}\big(\hat{\varphi}_{\boldsymbol{s},k,i} - \hat{\varphi}_{\boldsymbol{s},k}\big), \text{ and } \hat{\varphi}_{\boldsymbol{s}}^{(b)} = n^{-1}\sum_{k=1}^{K}\hat{\varphi}_{\boldsymbol{s},k}^{(b)}\cdot|\mathcal{I}_k|.$$

We use the joint distribution $\{\hat{\psi}_{\boldsymbol{s}}^{(b)}, \hat{\varphi}_{\boldsymbol{s}}^{(b)}\}_{b=1}^B$ to estimate the variance matrix of the orthogonal scores denoted by $\hat{\boldsymbol{V}}(\hat{\psi}_{\boldsymbol{s}}, \hat{\varphi}_{\boldsymbol{s}})$. We compute the standard error for the ratio $\hat{\varphi}_{\boldsymbol{s}}/\hat{\psi}_{\boldsymbol{s}}$ using the Delta method, namely, $\hat{\sigma} = \big(n^{-1}\boldsymbol{\omega}'\hat{\boldsymbol{V}}(\hat{\psi}_{\boldsymbol{s}},\hat{\varphi}_{\boldsymbol{s}})\boldsymbol{\omega}\big)^{1/2}$ where $\boldsymbol{\omega} = [-(\hat{\varphi}_{\boldsymbol{s}}/\hat{\psi}_{\boldsymbol{s}}^2), 1/\hat{\psi}_{\boldsymbol{s}}]'$.

The steps above differ from the estimation of type probabilities in a few instances. Step 2 uses the function $\nu_{\boldsymbol{s},\boldsymbol{t}}(Z)$ instead of $\nu_{\boldsymbol{s}}(T,Z)$. Steps 3 computes an additional parameter $\boldsymbol{\theta}$ while Step 4 computes two orthogonal scores. Steps 5 states that our estimator is a ratio of orthogonal scores means and Step 6 uses bootstrap and the delta method to evaluate the standard error of the ratio.

A.16 Doubly Robust Estimation Algorithm for Counterfactuals Using Comparison Compliers

We first consider the task of evaluating $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12})$. Steps 1, 5 and 6 of the previous procedure remain the same. Steps 2–4 are modified as following.

Step 2'. For each k, compute the estimator $\hat{\gamma}_{t,k,\boldsymbol{s}}$ as:

$$\hat{\boldsymbol{\gamma}}_{1,k} \in \arg\min_{\boldsymbol{\gamma}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(\frac{1}{2}(\boldsymbol{h}(Z_i, X_i)'\boldsymbol{\gamma})^2 + -(\boldsymbol{h}(1, X_i)' - \boldsymbol{h}(0, X_i))'\boldsymbol{\gamma}\right) + \alpha_{\boldsymbol{\gamma}}\|\boldsymbol{\gamma}\|_1. \tag{82}$$

Step 3'. For each partition k, compute the parameters $\hat{\boldsymbol{\beta}}_{1,k}$, $\hat{\boldsymbol{\beta}}_{2,k}$, $\hat{\boldsymbol{\theta}}_{1,k}$, $\hat{\boldsymbol{\pi}}_{1,k}$, and $\hat{\boldsymbol{\pi}}_{2,k}$, via the following lasso estimations:

$$\begin{aligned}
\hat{\boldsymbol{\theta}}_{1,k} &\in \arg\min_{\boldsymbol{\theta}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(Y \cdot 1[T_i = 1] - \boldsymbol{h}(Z_i, X_i)'\boldsymbol{\theta}\right)^2 + \alpha_{\boldsymbol{\theta}}\|\boldsymbol{\theta}\|_1,\\
\hat{\boldsymbol{\beta}}_{1,k} &\in \arg\min_{\boldsymbol{\beta}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(1[T_i = 1] - \boldsymbol{h}(Z_i, X_i)'\boldsymbol{\beta}\right)^2 + \alpha_{\boldsymbol{\beta},1}\|\boldsymbol{\beta}\|_1,\\
\hat{\boldsymbol{\beta}}_{2,k} &\in \arg\min_{\boldsymbol{\beta}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(1[T_i = 2] - \boldsymbol{h}(Z_i, X_i)'\boldsymbol{\beta}\right)^2 + \alpha_{\boldsymbol{\beta},\mathbf{2}}\|\boldsymbol{\beta}\|_1,\\
\hat{\boldsymbol{\pi}}_{1,k} &\in \arg\min_{\boldsymbol{\pi}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(1[T_i = 1]\boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\gamma}}_{1,k} - \boldsymbol{g}(X_i)'\boldsymbol{\pi}\right)^2 + \alpha_{\boldsymbol{\pi},1}\|\boldsymbol{\pi}\|_1,\\
\hat{\boldsymbol{\pi}}_{2,k} &\in \arg\min_{\boldsymbol{\pi}\in\mathbf{R}^p} \sum_{i\in\mathcal{I}_k^c} \left(1[T_i = 2]\boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\gamma}}_{1,k} - \boldsymbol{f}(X_i)'\boldsymbol{\pi}\right)^2 + \alpha_{\boldsymbol{\pi},2}\|\boldsymbol{\pi}\|_1,
\end{aligned}$$

where $\boldsymbol{f}(X) \equiv (f_1(X), \ldots, f_q(X))'$ denote a q-dimensional vector of functions of baseline variable.

Step 4'. Given $\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k}, \hat{\boldsymbol{\beta}}_{t,k}$, and $\hat{\boldsymbol{\theta}}_{t,k}$, we can compute the orthogonal score $\hat{\psi}_{\boldsymbol{s},i,k}$ regarding $P(\boldsymbol{S} = \boldsymbol{s}_{21})$ for each agent $i \in \mathcal{I}_k$ and each partition k :

$$\hat{\psi}_{\boldsymbol{s},k,i} \equiv \left(\boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\gamma}}_{\boldsymbol{s},t,k} \cdot \left(1[T_i = 2] - \boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\beta}}_{t,k}\right) + \boldsymbol{h}(1, X_i)' - \boldsymbol{h}(0, X_i)'\hat{\boldsymbol{\beta}}_{t,k}\right).$$

The orthogonal score for $E(Y(1)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}_{21}])$ is cumbersome. We define the following terms to facilitate notation: $\Theta_i \equiv \boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\theta}}_{1,k}$, $\Lambda_{1,i} \equiv \boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\beta}}_{1,k}$, $\Lambda_{2,i} \equiv \boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\beta}}_{2,k}$, $\Delta_i \equiv \boldsymbol{h}(0, X_i)' - \boldsymbol{h}(1, X_i)$, $\kappa_i \equiv \boldsymbol{h}(Z_i, X_i)'\hat{\boldsymbol{\gamma}}_{1,k}$. $U_i \equiv \boldsymbol{f}(X_i)'\hat{\boldsymbol{\pi}}_{1,k}$, and $C_i \equiv \boldsymbol{f}(X_i)'\hat{\boldsymbol{\pi}}_{2,k}/U_i$. In this notation, we can define the orthogonal score for $E(Y(1)\mathbf{1}[\boldsymbol{S} = \boldsymbol{s}_{21}])$ associated to agent $i \in \mathcal{I}_k$ and each partition k as:

$$\begin{aligned}
\hat{\varphi}_{\boldsymbol{s},k,i} \equiv{}& ((Y_i. * 1[T_i = 1] - \Theta_i)\kappa_i + (\Delta_i\Theta_i))C_i - (((1[T_i = 2] - \Lambda_{2,i})\kappa_i) \cdot (\Delta_i\Theta_i))\frac{1}{U_i}\\
&- (((1[T_i = 1] - \Lambda_{1,i})\kappa_i) \cdot (\Delta_i\Theta_i))\frac{C}{U_i} - (((\Delta_i\Lambda_{2,i})) \cdot (\Delta_i\Theta_i))\frac{1}{U_i} - (((\Delta_i\Lambda_{1,i})) \cdot (\Delta_i\Theta_i))\frac{C}{U_i}.
\end{aligned}$$

As mentioned, the Steps 5–6 remains the same. This estimator evaluates $E(Y(1)|\boldsymbol{S} = \boldsymbol{s}_{12})$ which enable us to estimate the causal effect $E(Y(2)-Y(1)|\boldsymbol{S}=\boldsymbol{s}_{12})$ since $E(Y(2)|\boldsymbol{S}=\boldsymbol{s}_{12})$ was already estimated. The standard error of the causal effect is obtained via the multiplier bootstrap. The counterfactual outcome $E(Y(1)|\boldsymbol{S}=\boldsymbol{s}_{13})$ is obtained by replacing the choice 2 in Steps 3' and Step 4' by the treatment choice 3.